Mathematical Modeling of Turbulent Flows

Prof. Dr.-Ing. habil. Nikolai Kornev

Rostock
2013
# Contents

1 Main Equations of Fluid Dynamics ........................................... 13
   1.1 Continuity equation ................................................. 13
   1.2 Classification of forces acting in a fluid. ......................... 14
      1.2.1 Body forces ............................................... 14
      1.2.2 Surface forces. ........................................... 15
      1.2.3 Properties of surface forces. .............................. 15
   1.3 Navier Stokes Equation .............................................. 17

2 Physics of turbulence ....................................................... 23
   2.1 Definition of the turbulence ....................................... 23
   2.2 Vortex dynamics .................................................... 23
      2.2.1 Vorticity transport equation ............................... 23
      2.2.2 Vorticity and vortices ..................................... 24
      2.2.3 Vortex amplification as an important mechanism of the
turbulence generation ............................................. 26
      2.2.4 Vortex reconnection ......................................... 28
      2.2.5 Richardson poem (1922) .................................... 29
      2.2.6 Summary .................................................... 30
   2.3 Experimental observations ......................................... 30
      2.3.1 Laminar- turbulent transition in pipe. Experiment of
      Reynolds. ......................................................... 31
      2.3.2 Laminar- turbulent transition and turbulence in jets. . 33
      2.3.3 Laminar- turbulent transition in wall bounded flows. . 36
      2.3.4 Uneven distribution of the vorticity in the turbulent
      flows at large Reynolds number ............................... 37
      2.3.5 Distribution of the averaged velocity in the turbulent
      boundary layer ................................................. 42

3 Basic definitions of the statistical theory of turbulence ............ 49
   3.1 Reynolds averaging ............................................... 49
   3.2 Isotropic and homogeneous turbulence ............................ 50
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td>Correlation function. Integral length.</td>
<td>50</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Some relations in isotropic turbulence</td>
<td>53</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Taylor microscale λ</td>
<td>55</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Correlation functions in the Fourier space</td>
<td>56</td>
</tr>
<tr>
<td>3.3.4</td>
<td>Spectral density of the kinetic energy</td>
<td>57</td>
</tr>
<tr>
<td>3.4</td>
<td>Structure functions</td>
<td>57</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Probability density function</td>
<td>57</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Structure function</td>
<td>58</td>
</tr>
<tr>
<td>4</td>
<td>Kolmogorov theory K41</td>
<td>61</td>
</tr>
<tr>
<td>4.1</td>
<td>Physical background</td>
<td>61</td>
</tr>
<tr>
<td>4.2</td>
<td>Dissipation rate</td>
<td>63</td>
</tr>
<tr>
<td>4.3</td>
<td>Kolmogorov hypotheses</td>
<td>64</td>
</tr>
<tr>
<td>4.4</td>
<td>Three different scale ranges of turbulent flow</td>
<td>66</td>
</tr>
<tr>
<td>4.5</td>
<td>Classification of methods for calculation of turbulent flows</td>
<td>69</td>
</tr>
<tr>
<td>4.6</td>
<td>Limitation of K-41. Kolmogorov theory K-62</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>Reynolds Averaged Navier Stokes Equation (RANS)</td>
<td>73</td>
</tr>
<tr>
<td>6</td>
<td>Reynolds Stress Model (RSM)</td>
<td>79</td>
</tr>
<tr>
<td>6.1</td>
<td>Derivation of the RSM Equations</td>
<td>79</td>
</tr>
<tr>
<td>6.1.1</td>
<td>Step 1</td>
<td>79</td>
</tr>
<tr>
<td>6.1.2</td>
<td>Step 2</td>
<td>80</td>
</tr>
<tr>
<td>6.1.3</td>
<td>Step 3</td>
<td>80</td>
</tr>
<tr>
<td>6.1.4</td>
<td>Analysis of terms</td>
<td>82</td>
</tr>
<tr>
<td>7</td>
<td>Equations of the k - ε Model</td>
<td>85</td>
</tr>
<tr>
<td>7.1</td>
<td>Derivation of the k-Equation</td>
<td>85</td>
</tr>
<tr>
<td>7.1.1</td>
<td>Closure of terms of $k$ equation</td>
<td>86</td>
</tr>
<tr>
<td>7.1.2</td>
<td>Derivation of the ε-Equation</td>
<td>87</td>
</tr>
<tr>
<td>8</td>
<td>Large Eddy Simulation (LES)</td>
<td>89</td>
</tr>
<tr>
<td>8.1</td>
<td>LES filtering</td>
<td>89</td>
</tr>
<tr>
<td>8.1.1</td>
<td>Properties of filtering</td>
<td>90</td>
</tr>
<tr>
<td>8.2</td>
<td>LES equations</td>
<td>91</td>
</tr>
<tr>
<td>8.3</td>
<td>Smagorinsky model</td>
<td>92</td>
</tr>
<tr>
<td>9</td>
<td>Subgrid Stress (SGS) Models</td>
<td>95</td>
</tr>
<tr>
<td>9.1</td>
<td>Model of Germano (Dynamic Smagorinsky Model)</td>
<td>95</td>
</tr>
<tr>
<td>9.2</td>
<td>Scale similarity models</td>
<td>97</td>
</tr>
<tr>
<td>9.3</td>
<td>Mixed similarity models</td>
<td>98</td>
</tr>
<tr>
<td>9.3.1</td>
<td>A-posteriori and a-priori tests</td>
<td>100</td>
</tr>
</tbody>
</table>
10 Hybrid URANS-LES methods

10.1 Introduction .............................................. 103
10.2 Detached Eddy Simulation (DES) ...................... 104
10.3 Description of the hybrid model proposed in Rostock ...... 107
10.4 Sample of hybrid method application for the tanker KVLCC2 110
   10.4.1 Validation ............................................ 113
List of Tables

8.1 Properties of large and small scale motions . . . . . . . . . . . 91
8.2 Advantages and disadvantages of the Smagorinsky model . . . 94

10.1 Results of the resistance prediction using different methods. 
$C_R$ is the resistance coefficient, $C_P$ is the pressure resistance
and $C_F$ is the friction resistance. . . . . . . . . . . . . . . . . . . 113
List of Figures

1.1 Body and surface forces acting on the liquid element. .......... 14
1.2 Forces acting on the liquid element. .......................... 16
1.3 Stresses acting on the liquid cube with sizes $a$. ............... 17

2.1 Vorticity and vortices ........................................... 25
2.2 Tornado ..................................................................... 25
2.3 Vortices in two-dimensional and three dimensional cases. .... 26
2.4 Velocities induced by vortices. Three dimensional curvilinear
vortices induce self induced velocities. ............................. 26
2.5 Illustration of the vortex folding. ................................. 28
2.6 Scenario of vortex amplification ................................. 28
2.7 Scenario of vortex reconnection ................................. 29
2.8 Sample of the vortex reconnection of tip vortices behind an
airplane. ...................................................................... 30
2.9 Most outstanding results in turbulence research according to
[1]. ............................................................................. 31
2.10 Sketch of the Reynolds experiment. ............................. 32
2.11 Development of instability during the laminar- turbulent tran-
sition in the circular pipe (taken from [1]). ........................ 32
2.12 Development of instability in the jet (taken from [1]). ....... 33
2.13 Development of instability in the free jet. ....................... 34
2.14 Development of instability in the free jet. ....................... 34
2.15 Vortex structures in a free jet in a far field. .................... 34
2.16 Vortex structures in a free jet with acoustic impact. .......... 35
2.17 Vortex structures in a confined jet mixer flow. ................. 36
2.18 Fine vortex structures in a confined jet mixer flow. PLIF mea-
surements by Valery Zhdanov (LTT Rostock). Spatial resolu-
tion is 31$\mu$m. .......................................................... 37
2.19 Scenario of laminar turbulent transition in the boundary layer
on a flat plate. ......................................................... 38
2.20 Streaks visualized by hydrogen bubbles in the boundary layer
on a flat plate. ......................................................... 38
2.21 Conceptual model of the organization of the turbulence close to the wall proposed by Adrian et al. (2000).

2.22 Sketch of the flow. 1- knee bend of nozzle, 2- plate for damping of vortices shed from knee bend 1, 3- outer tube, 4- support plates, 5- nozzle, 6- test section, 7- water box.

2.23 Snapshot of the field $\omega_z^2 < \omega_z^2 >$ within the measurement window in jet mixer. The averaged $< \omega_z^2 >$ was $1.19s^{-2}$ and $0.459s^{-2}$ at, respectively, $x/D = 1$ and 7.

2.24 Snapshot of the field $(u_xu_x + u_yu_y)/ < u_xu_x + u_yu_y >$ within the measurement window in jet mixer. The averaged $< u_xu_x + u_yu_y >$ was $0.011(m/s)^2$ and $0.0024(m/s)^2$ at, respectively, $x/D = 1$ and 7.

2.25 Ratios $\varepsilon_k$ and $E_k$ depending on $k/N$.

2.26 Influence of the laser thickness on $\varepsilon_k$.

2.27 Probability density function of radius of structures of the field $\omega_z^2$.

2.28 Probability density function of the axis ratio $a/b \sqrt{ab}$ of structures of the field $\omega_z^2$.

2.29 Conditioned probability density function of the axis ratio $a/b \sqrt{ab}(R > m\Delta)$ of structures of the field $\omega_z^2$.

2.30 Vertical distribution of the velocity $u_x$ at three different time instants in boundary layer.

2.31 Illustration of the Prandtl derivation.

2.32 Structure of the velocity distribution in the turbulent boundary layer. $U^+ = u_x/u_r$.

3.1 Autocorrelation function coefficient for scalar fluctuation at three different points $A, B$ and $C$ across the jet mixer.

3.2 Distribution of the integral length of the scalar field along the jet mixer centerline.

3.3 Autocorrelation functions in free jet flow.

3.4 Illustrations of velocities used in calculations of the longitudinal $f$ and transversal $g$ autocorrelations.

3.5 Illustration of the autocorrelation functions $f$ and $g$ and Taylor microscales.

3.6 Kurtosis of the structure function for the concentration of the scalar field obtained in the jet mixer.
4.1 Andrey Kolmogorov was a mathematician, preeminent in the 20th century, who advanced various scientific fields (among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity).

4.2 Illustration of the vortex cascado

4.3 Turbulent vortices revealed in DNS calculations performed by Isazawa et al. (2007)

4.4 Distribution of the Kolmogorov scale along the centerline of the jet mixer and free jet. The dissipation rate $\varepsilon$ is calculated from the $k-\varepsilon$ model and the experimental estimation of Miller and Dimotakis (1991) $\varepsilon = 48(U_d^3/d)((x-x_0)/d)^{-4}$

4.5 Three typical scale ranges in the turbulent flow at high Reynolds numbers

4.6 Three typical ranges of the energy density spectrum in the turbulent flow at high Reynolds number. 1- energy containing range, 2- inertial subrange, 3- dissipation range.

4.7 Experimental confirmation of the Kolmogorov law. The compensated energy spectrum for different flows.

4.8 Experimental confirmation of the Kolmogorov law for the concentration fluctuations in the jet mixer. Measurements of the LITT Rostock.

4.9 Three main methods of turbulent flows modelling.

4.10 Vortex structures resolved by different models

4.11 Power of the structure function. Experiments versus prediction of Kolmogorov and Obukhov

8.1 Different filtering functions used in LES

9.1 Illustrations for derivation of the scale similarity model

10.1 Zones of the Detached Eddy Simulation

10.2 Squires K.D., Detached-eddy simulation: current status and perspectives

10.3 Squires K.D., Detached-eddy simulation: current status and perspectives

10.4 The division of the computational domain into the URANS (dark) and LES (light) regions at one time instant.

10.5 The cell parameters.

10.6 The mean axial velocity field $u_x/u_0$ in the propeller plane computed with different models (right) vs. measurements (left).
10.7 Circumferential distribution of the mean axial velocity field in
the propeller plane. △ — $k$-$\omega$-SST-SAS, ○ — DSM+V2F,
solid line — KRISO experiments for the specified $r/R$. . . . . 115
10.8 Resolved turbulent kinetic energy $k = \rho/2(u'_x u'_x + u'_y u'_y + $
$u'_z u'_z)/u_0^2$ multiplied with $10^3$ in the propeller plane. Numerics
(right-half of each figure) versus measurement (left-half). . . 116
10.9 Positions of probe points. $R$ is the propeller radius. . . . . . . . 116
10.10 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 117
Chapter 1

Main Equations of Fluid Dynamics

1.1 Continuity equation

We consider the case of uniform density distribution \( \rho = \text{const} \). The continuity equation has the following physical meaning: The amount of liquid flowing into the volume \( U \) with the surface \( S \) is equal to the amount of liquid flowing out. Mathematically it can be expressed in form:

\[
\int_S \vec{u} \cdot \vec{n} \, ds = 0
\] (1.1)

Expressing the scalar product \( \vec{u} \cdot \vec{n} \)

\[
\int_S \left( u_x \cos(nx) + u_y \cos(ny) + u_z \cos(nz) \right) \, ds = 0.
\]

and the Gauss theorem we get

\[
\int_U \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \, dU = 0
\]

Since the integration volume \( U \) is arbitrary, the integral is zero only if

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0
\] (1.2)

In the tensor form the continuity equation reads:
\[
\frac{\partial u_i}{\partial x_i} = 0 \quad \text{(1.3)}
\]

1.2 Classification of forces acting in a fluid.

The inner forces acting in a fluid are subdivided into the body forces and surface forces (Fig. 8.1).

![Figure 1.1: Body and surface forces acting on the liquid element.](image)

1.2.1 Body forces

Let \( \Delta f \) be a total body force (integral of \( f \)) acting on the volume \( \Delta U \). Let us introduce the strength of the body force as limit of the ratio of the force to the volume:

\[
\vec{F} = \lim_{\Delta U \to 0} \frac{\Delta \vec{f}}{\rho \Delta U} \quad \text{(1.4)}
\]
which has the unit \( \frac{kg \, m^3}{s^2} = ms^{-2} \). Typical body forces are gravitational, electrostatic or electromagnetic forces. For instance, we have the following relations for the gravitational forces:

\[
\Delta \vec{f} = \rho g \Delta \vec{U} \hat{k}
\]  

(1.5)

where \( \Delta \vec{f} \) is the gravitational force acting on a particle with volume \( \Delta U \). The strength of the gravitational force is equal to the gravitational acceleration:

\[
\vec{F} = \lim_{\Delta U \to 0} \left( -\frac{\rho g \Delta \vec{U} \hat{k}}{\rho \Delta U} \right) = -g \hat{k}
\]  

(1.6)

The body forces are acting at each point of fluid in the whole domain.

1.2.2 Surface forces.

The body forces are acting at each point at the boundary of the fluid element. Usually they are shear and normal stresses. The strength of surface forces is determined as

\[
\vec{p}_n = \lim_{\Delta S \to 0} \frac{\Delta \vec{P}_n}{\Delta S}
\]  

(1.7)

with the unit \( \frac{kg \, m^3}{s^2} = \frac{kg}{ms^2} \). A substantial feature of the surface force is the dependence of \( \vec{p}_n \) on the orientation of the surface \( \Delta S \).

The surface forces are very important because they act on the body from the side of liquid and determine the forces \( \vec{R} \) arising on bodies moving in the fluid:

\[
\vec{R} = \int_S \vec{p}_n dS
\]

\[
\vec{M} = \int_S (\vec{r} \times \vec{p}_n) dS
\]  

(1.8)

1.2.3 Properties of surface forces.

Let us consider a liquid element in form of the tetrahedron (Fig. 1.2). Its motion is described by the 2nd law of Newton:

\[
\rho \Delta U \frac{d\vec{u}}{dt} = \rho \Delta U \vec{F} + \vec{p}_n \Delta S - \vec{p}_x \Delta S_x - \vec{p}_y \Delta S_y - \vec{p}_z \Delta S_z
\]  

(1.9)
Dividing r.h.s and l.h.s. by the surface of inclined face $\Delta S$ results in:

$$\rho \frac{\Delta U}{\Delta S} \left( \frac{d\vec{u}}{dt} - \vec{F} \right) = \vec{p}_n - \vec{p}_x \frac{\Delta S_x}{\Delta S} - \vec{p}_y \frac{\Delta S_y}{\Delta S} - \vec{p}_z \frac{\Delta S_z}{\Delta S} \quad (1.10)$$

Let us find the limit of (1.10) at $\Delta S \to 0$:

$$\lim_{\Delta S \to 0} \frac{\Delta U}{\Delta S} = 0, \quad \lim_{\Delta S \to 0} \frac{\Delta S_x}{\Delta S} = \cos(nx), \quad (1.11)$$

$$\lim_{\Delta S \to 0} \frac{\Delta S_y}{\Delta S} = \cos(ny), \quad \lim_{\Delta S \to 0} \frac{\Delta S_z}{\Delta S} = \cos(nz) \quad (1.12)$$

Substitution of (1.11) and (1.12) into (1.10) results in the following relation between $\vec{p}_n$ and $\vec{p}_x, \vec{p}_y, \vec{p}_z$:

$$\vec{p}_n = \vec{p}_x \cos(nx) + \vec{p}_y \cos(ny) + \vec{p}_z \cos(nz) \quad (1.13)$$

Let us write the surface forces through components:

$$\vec{p}_x = \vec{i}p_{xx} + \vec{j}p_{xy} + \vec{k}p_{xz}$$

$$\vec{p}_y = \vec{i}p_{yx} + \vec{j}p_{yy} + \vec{k}p_{yz}$$

$$\vec{p}_z = \vec{i}p_{zx} + \vec{j}p_{zy} + \vec{k}p_{zz}$$
Here $\tau_{ij}$ are shear stress (for instance $\tau_{12} = \tau_{xy}$), whereas $p_{ii}$ are normal stress (for instance $p_{11} = p_{xx}$). From moment equations (see Fig. 1.3) one can obtain the symmetry condition for shear stresses: $\tau_{xy}a - \tau_{yz}a = 0 \Rightarrow \tau_{xy} = \tau_{yz}$, and generally:

$$\tau_{ij} = \tau_{ji} \quad (1.14)$$

The stress matrix is symmetric and contains 6 unknown elements:

$$
\begin{pmatrix}
p_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & p_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & p_{zz}
\end{pmatrix}
$$

(1.15)

1.3 Navier Stokes Equation

Applying the Newton second law to the small fluid element $dU$ with the surface $dS$ and using the body and surface forces we get:

$$\int \frac{d\vec{U}}{dt} \rho dU = \int \vec{F} \rho dU + \int \vec{p}_n dS \quad (1.16)$$

The property of the surface force can be rewritten with the Gauss theorem in the following form:
\[ \int_{S} \vec{p}_{n} dS = \int_{S} \left( \vec{p}_{x} \cos(nx) + \vec{p}_{y} \cos(ny) + \vec{p}_{z} \cos(nz) \right) dS \]
\[ = \int_{U} \left( \frac{\partial \vec{p}_{x}}{\partial x} + \frac{\partial \vec{p}_{y}}{\partial y} + \frac{\partial \vec{p}_{z}}{\partial z} \right) dU \]

The second law (1.16) takes the form:
\[ \int_{U} \frac{d\vec{u}}{dt} \rho dU = \int_{U} \vec{F} \rho dU + \int_{U} \left( \frac{\partial \vec{p}_{x}}{\partial x} + \frac{\partial \vec{p}_{y}}{\partial y} + \frac{\partial \vec{p}_{z}}{\partial z} \right) dU = \int_{U} \left[ \frac{d\vec{u}}{dt} \rho - \rho \vec{F} - \left( \frac{\partial \vec{p}_{x}}{\partial x} + \frac{\partial \vec{p}_{y}}{\partial y} + \frac{\partial \vec{p}_{z}}{\partial z} \right) \right] dU = 0 \]

Since the volume \( dU \) is arbitrary, the r.h.s. in the last formulae is zero only if:
\[ \frac{d\vec{u}}{dt} = \vec{F} + \frac{1}{\rho} \left( \frac{\partial \vec{p}_{x}}{\partial x} + \frac{\partial \vec{p}_{y}}{\partial y} + \frac{\partial \vec{p}_{z}}{\partial z} \right) \]  \hspace{1cm} (1.17)

The stresses in (1.17) are not known. They can be found from the generalized Newton hypothesis:
\[ \begin{pmatrix} p_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & p_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & p_{zz} \end{pmatrix} = - \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + 2\mu S_{ij} \]  \hspace{1cm} (1.18)

where \( p \) is the pressure,
\[ S_{11} = S_{xx} = \frac{\partial u_{x}}{\partial x}; \quad S_{12} = S_{xy} = \frac{1}{2} \left( \frac{\partial u_{x}}{\partial y} + \frac{\partial u_{y}}{\partial x} \right); \quad S_{13} = S_{xz} = \frac{1}{2} \left( \frac{\partial u_{x}}{\partial z} + \frac{\partial u_{z}}{\partial x} \right) \]
\[ S_{21} = S_{12}, \quad S_{22} = S_{yy} = \frac{\partial u_{y}}{\partial y}; \quad S_{23} = S_{yz} = \frac{1}{2} \left( \frac{\partial u_{y}}{\partial z} + \frac{\partial u_{z}}{\partial y} \right) \]
\[ S_{31} = S_{13}, \quad S_{32} = S_{23}, \quad S_{33} = S_{zz} = \frac{\partial u_{z}}{\partial z} \]

The liquids obeying (1.18) are referred to as the Newtonian liquids. The normal stresses can be expressed through the pressure \( p \):
\[ p_{xx} = -p + 2\mu \frac{\partial u_x}{\partial x}, \quad p_{yy} = -p + 2\mu \frac{\partial u_y}{\partial y}, \quad p_{zz} = -p + 2\mu \frac{\partial u_z}{\partial z} \]

The sum of three normal stresses doesn’t depend on the choice of the coordinate system and is equal to the pressure taken with sign minus:

\[ \frac{p_{xx} + p_{yy} + p_{zz}}{3} = -p \]  

(1.19)

The last expression is the definition of the pressure in the viscous flow: The pressure is the sum of three normal stresses taken with the sign minus. Substitution of the Newton hypothesis (1.18) into (1.17) gives (using the first equation as a sample):

\[ \rho \frac{du_x}{dt} = \rho F_x + \frac{\partial}{\partial x} \left( -p + 2\mu \frac{\partial u_x}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) \right) \]

\[ + \frac{\partial}{\partial z} \left( \mu \left( \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \right) = \]

\[ = \rho F_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + \]

\[ + \mu \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \]

The last term in the last formula is zero because of the continuity equation. Doing similar transformation with resting two equations in \( y \) and \( z \) directions, one can obtain the following equation, referred to as the Navier-Stokes equation:

\[ \frac{d\vec{u}}{dt} = \vec{F} - \frac{1}{\rho} \nabla p + \nu \Delta \vec{u} \]  

(1.20)

The full or material substantial derivative of the velocity vector \( \frac{d\vec{u}}{dt} \) is the acceleration of the fluid particle. It consists of two parts: local acceleration and convective acceleration:

\[ \frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + u_x \frac{\partial \vec{u}}{\partial x} + u_y \frac{\partial \vec{u}}{\partial y} + u_z \frac{\partial \vec{u}}{\partial z} \]

local acceleration  convective acceleration
The local acceleration is due to the change of the velocity in time. It is present even if the particle is at the rest. The convective acceleration is due to particle motion in a nonuniform velocity field. The Navier-Stokes Equation in tensor form is:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = F_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i \right) \tag{1.21}$$

Using the continuity equation (1.3) the convective term can be written in the conservative form:

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( u_i u_j \right) \tag{1.22}$$

Finally, the Navier Stokes in the tensor form is:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = F_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i \right) \tag{1.23}$$

The Navier Stokes equation together with the continuity equation (1.3) is the closed system of partial differential equations. Four unknowns velocity components $u_x, u_y, u_z$ and pressure $p$ are found from four equations. The equation due to presence of the term $\frac{\partial}{\partial x_j} (u_i u_j)$ is nonlinear.

The boundary conditions are enforced for velocity components and pressure at the boundary of the computational domain. The no slip condition $u_x = u_y = u_z = 0$ is enforced at the solid body boundary. The boundary condition for the pressure at the body surface can directly be derived from the Navier Stokes equation. For instance, if $y = 0$ corresponds to the wall, the Navier Stokes Equation takes the form at the boundary:

$$\frac{\partial p}{\partial x} = \rho F_x + \mu \frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = \rho F_y + \mu \frac{\partial^2 u_y}{\partial y^2}$$

$$\frac{\partial p}{\partial z} = \rho F_z + \mu \frac{\partial^2 u_z}{\partial y^2}$$

Very often the last term in the last formulae is neglected because second spatial derivatives of the velocity are not known at the wall boundary.
Till now, the existence of the solution of Navier Stokes has been not proven by mathematicians. Also, it is not clear whether the solution is smooth or allows singularity. The Clay Mathematics Institute has called the Navier–Stokes existence and smoothness problems one of the seven most important open problems in mathematics and has offered one million dollar prize for its solution.
Chapter 2

Physics of turbulence

2.1 Definition of the turbulence

Flow motions are subdivided into laminar flows and turbulent ones. The word "Laminar" in Greek means layer. The fluid particles move orderly in layers without intense lateral mixing. The disruption between layers is absent. On the contrary the turbulent flow is very chaotic with strong eddies and intense mixing across the flow.

Turbulent motion is the three dimensional unsteady flow motion with
• chaotical trajectories of fluid particles,
• fluctuations of the velocity and
• strong mixing
arisen at large Re numbers due to unstable vortex dynamics.

2.2 Vortex dynamics

The vortex dynamics is the key to understand what happens in the turbulent flow.

2.2.1 Vorticity transport equation

The vector calculus relation reads:

\[ \frac{1}{2} \nabla (A \cdot A) = A \times (\nabla \times A) + (A \nabla)A \]  (2.1)

Taking \( u = A \) we get:

\[ \frac{1}{2} \nabla (u \cdot u) = u \times \omega + (u \nabla)u \]  (2.2)
where $\omega = \nabla \times \mathbf{u}$ is the vorticity. Here we used the identity $\nabla \times (\frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u})) = 0$. Application of the curl operator to (2.3) results in

$$\nabla \times ((\mathbf{u} \nabla) \mathbf{u}) = -\nabla \times (\mathbf{u} \times \omega) = -\mathbf{u}(\nabla \omega) + \omega(\nabla \mathbf{u}) - (\omega \nabla) \mathbf{u} + (\mathbf{u} \nabla) \omega$$  \hspace{1cm} (2.3)

Both vectors $\mathbf{u}$ and $\omega$ satisfy the continuity equation, i.e. $\nabla \mathbf{u} = 0$ and $\nabla \omega = 0$:

$$\nabla \times ((\mathbf{u} \nabla) \mathbf{u}) = -\omega(\nabla) \mathbf{u} + (\mathbf{u} \nabla) \omega$$  \hspace{1cm} (2.4)

Let us apply the curl operator to the Navier Stokes equation

$$\nabla \times (\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u}) = \nabla \times (-\frac{1}{\rho} \nabla \mathbf{p} + \nu \Delta \mathbf{u})$$  \hspace{1cm} (2.5)

$$\frac{\partial \omega}{\partial t} + \nabla \times ((\mathbf{u} \nabla) \mathbf{u}) = \nu \Delta(\nabla \times \mathbf{u}) = \nu \Delta \omega$$  \hspace{1cm} (2.6)

Substituting (2.4) into (2.6) results in

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \nabla) \omega = (\omega \nabla) \mathbf{u} + \nu \Delta \omega$$  \hspace{1cm} (2.7)

$$\frac{D \omega}{Dt} = (\omega \nabla) \mathbf{u} + \nu \Delta \omega$$  \hspace{1cm} (2.8)

The equation (2.8) is the vorticity transport equation.

### 2.2.2 Vorticity and vortices

The vortices are main players in turbulent flows. Here we would like to emphasize the difference between the vorticity and vortices. The vorticity is the curl of the velocity $\omega = \nabla \times \mathbf{u}$. The vorticity is usually not zero in viscous flows especially in areas close to the walls. Speaking about vortices we bear in mind the concentrated structures of the vorticity field $\omega = \nabla \times \mathbf{u}$.

The difference between the vorticity and vortices is illustrated in Fig. 2.1. The boundary layer is the flow area with strong but smoothly distributed vorticity (Fig. 2.1a). Due to instabilities, that will be discussed later, the concentrated vortex structures arise in the smooth vorticity field (Fig. 2.1b). A famous sample of concentrated vortex structures is the tornado (Fig. 2.2).

The vorticity is solenoidal:

$$\nabla \omega = \nabla(\nabla \times \mathbf{u}) = 0$$  \hspace{1cm} (2.9)

The consequence of the condition (2.9) is:

- All vortex lines, defined as the lines which are tangential to the vorticity vector $\omega \times dl = 0$, are closed in the three dimensional case (Figure 2.3).
The velocity induced by vorticity $\omega$ occupied the volume $U$ are calculated from the Biot-Savart law:

$$u(x, t) = \frac{1}{4\pi} \int_U \frac{\omega \times (x - r)}{|x - r|^3}$$  

(2.10)

The velocities induced by two dimensional and three dimensional vortex structures are shown in Fig. 2.4. An important fact is the appearance of self induced velocities on curvilinear three dimensional vortex structures. They are responsible for leapfrog vortex ring motion (http://www.lemos.uni-rostock.de/galerie/). The self induced velocities is the reason for convective instability of three dimensional vortex structures.

Figure 2.1: Vorticity and vortices

Figure 2.2: Tornado
2.2.3 Vortex amplification as an important mechanism of the turbulence generation

As mentioned above the vortices are main players during the laminar- turbulent transition. The vorticity can be essentially intensified (amplified) due to action of neighboring vortices or even due to self induction. The reason
for that can be explained by analysis of the vorticity transport equation

\[
\frac{D\omega}{Dt} = (\omega \nabla)u + \nu \Delta \omega
\]  

(2.11)

The r.h.s. of (2.11) contains two terms. The first term \((\omega \nabla)u\) is responsible for the rotation of the vorticity vector \(\omega\) and enlargement or reduction of its magnitude \(|\omega|\). The second diffusion term results in spreading of the vorticity in the space. The term \((\omega \nabla)u\) is responsible for the amplification of the vorticity.

The effect of the amplification can easily be understood if we consider the vortex with vector aligned along the \(x\)-axis \(\omega_x > 0\). If such a vortex is in the fluid stretching area \(\frac{\partial u_x}{\partial x} > 0\), the term \(\omega_x \frac{\partial u_x}{\partial x} > 0\) is positive. As a result, \(\frac{D\omega_x}{Dt} > 0\) is positive, what leads to the increase of the vorticity \(\omega_x\). As shown analytically by Novikov [2] for a simple model problem, the vortex strength of vortons structures can increase exponentially up to the infinity without viscosity effects. The vorticity growth caused by inviscid amplification term is counterbalanced by the diffusion term. Two terms on r.h.s. of (2.11) compete with each other. In the inviscid fluid the circulation of the vortex core is constant \(\Gamma = \int_S \omega_x dS = const\). Increase of \(\omega_x\) results in the decrease of the cross section \(S\). The vortex becomes thinner. The diffusion acts against and makes the vortex thicker. In some flow regions the amplification can be stronger than diffusion. The thin vortex losses the stability and is folded.

As shown by by Chorin [3] and [4] the folding is necessary mechanism preventing the exponential growth of vorticity. If the amplification is too strong, the vorticity goes to infinity and the energy is not kept constant. Chorin [3] notes that "as the vortices stretch, their cross-section decreases and the energy associated with them would increase unless they arranged themselves in such a way that their velocity canceled. The foldings achieves such cancelation". This could be easily explained using a simple sample. If we have just one straight infinite vortex it induces the velocity in the plane perpendicular to its axis. If this vortex is tangled the vortex pieces with different vorticity direction are approaching close each to other canceling their induction (see Fig. 2.5). Chorin [4] explicitly specifies typical scales of folding: "the inertial range \(^1\) properties are due to the appearance of folded vortex tubes, which behave on large scales as self-avoiding walks, and on small scales contain a large number of folds (=hairpins) that are needed to satisfy the constraint of energy conservation".

Summarizing all effects mentioned above one can imagine the following scenario (see Fig. 2.6). Let the vorticity at a certain point of the flow grows.

\(^1\) this term will be introduced in the next chapter
In the real physical process folding prevents the growth of the kinetic energy and increases the canceling effect of viscosity. Then the vortex structures breaks down into small structures due to reconnection mechanism described in the next subsection.

2.2.4 Vortex reconnection

Let us consider the vortex ring (see Figure (2.7)). Due to convective instability or influence of neighboring vortices the vortex ring is deformed. Due to self induction two opposite sides of the ring are merged. As soon as two elements with opposite vorticity sign are approaching each to other, they start to cancel each other by mutual diffusion. The vortices disappear in the area of the contact. Two small vortices are created from one big vortex. Each small vortex ring breaks then down into smaller vortices and so on.
The energy of small vortices is equal to the energy of the big vortex with a small loss caused by the dissipation. This fact is formulated in the sentence, which is common in the turbulence theory: The energy is transferred from large scale vortices to small scales vortices. The reconnection process can be observed on macroscales. The decay, break up of tip vortices behind the airplane proceeds according to the same scenario (see Fig. (2.8)). The reconnection process is, perhaps, the main mechanism of vortex cascade in the turbulent flows, i.e. transformation of big vortices into small ones. The reconnection can also lead to enlargement of small vortices if two rings approach each to other as shown in Fig. 2.7 (see red circle). In this case the energy of small vortices turns into the energy of the big vortex. This process is referred to as the energy back scattering. Statistically, the direct energy flux sufficiently surpasses the backward one.

2.2.5 Richardson poem (1922)

The vortex turbulence cascade means that large eddies break down to form small eddies as turbulence cascades from large scales to small ones. This idea was formulated in the famous poem by Richardson (1922):

Big whorls have little whorls,
Which feed on their velocity;
Figure 2.8: Sample of the vortex reconnection of tip vortices behind an airplane.

*And little whorls have lessor whorls,*

*And so on to viscosity (in the molecular sense).*

### 2.2.6 Summary

Vortex arise in the fluid due to viscosity effects. They experience instability and amplification. Diffusion counteracts the amplification. If the Reynolds number is large, the vortex structures are strong and concentrated. The amplification can dominate at some fluid region over the diffusion. The vortex instability is not damped by viscosity. The flow becomes stochastic due to mutual interaction of unstable vortices. The big vortices break down into small ones by means of vortex reconnection. The vortex instability process is identified as the turbulence.

### 2.3 Experimental observations

Different regimes of the fluid motion were revealed very early, perhaps, in antique times. Much later, Leonardo da Vinci recognized two states of the fluid motion and introduced the term "la turbolenza". Arkady Tsinober in his book "An informal introduction to turbulence" [1] presented most outstanding results in turbulence research in chronological order (Fig. 2.9).
2.3.1 Laminar- turbulent transition in pipe. Experiment of Reynolds.

Quantitative study of turbulence was started by Osborne Reynolds (1842-1912) who performed in 1883 very famous experiment shown in Fig. 2.10. The water flows from the vessel A to the pipe B. The ink injected into the pipe B with the local flow velocity is not mixed in transversal direction and keeps its identity if the flow velocity is small (Fig. 2.10, right). The flow under such a condition is laminar. As soon as the flow velocity increases due to water level raise in the vessel A, the ink jet loses the stability and is mixed with surrounding water (Fig. 2.10, right). The flow becomes turbulent. Ink jet development at different flow velocities in the circular pipe is shown in Fig. 2.11. The great merit of Reynolds lies in the fact, that he in contrast to his predecessors quantified the laminar turbulent transition. He showed that the transition in pipes occurs if the Reynolds number \( Re = U_b D/\nu \) exceeds the threshold around \( \sim 2400 \). Here \( U_b \) is the bulk velocity in pipe determined as the ratio of the flow rate to the pipe cross section, i.e. \( U_b = Q/(\pi D^2/4) \), \( D \) is the pipe diameter and \( \nu \) is the kinematic viscosity coefficient (\( \nu \sim 10^{-6}m^2s^{-1} \) for water and \( \nu \sim 10^{-6}m^2s^{-1} \) for air). Later, it was shown that the transition strongly depends on the perturbations presented in the flow. The experimental setup of Reynolds has been preserved at the University of Manchester in UK. The experiments done nowadays shown that the laminar-
turbulent transition Reynolds number is less than that documented originally by Reynolds. The reason is, presumably, the building vibration and noise caused by traffic which was not in time of Reynolds. If the perturbations are eliminated the transition can be delayed up to $Re \sim 40000...50000$. 

32
2.3.2 Laminar- turbulent transition and turbulence in jets.

The jet flows experiences also laminar turbulent transition shown in Fig. 2.12. Obviously, the flow can be fully or only partially turbulent. Close to the nozzle the flow is laminar. The instability is developed downstream. The shear flow at the jet boundary is the area of rapid velocity change from the jet velocity to zero outside of the jet. The shear flow experiences the so called Kelvin Helmholtz instability (Fig. 2.13) resulting in formation of concentrated vortices which are approximately circular. It happens close to jet nozzle at $x/D < 1$ (see Fig. 2.14), where $x$ is the distance from the nozzle and $D$ is the nozzle diameter.

The Kelvin Helmholtz vortices experience then the pairing (see Fig. 2.14). One vortex overtakes the neighbor vortex creating a pair. This process has an inviscid convective nature and can be explained thinking back to the famous leapfrog motion of two vortex rings. In the inviscid flow the leapfrog motion is running as long as the convective instability destroys the vortices. One vortex runs the next down, its radius decreases whereas the speed increases. The radius of the next vortex increases, the speed decreases. The first ring moves through the second one. The process is then repeated. The movie illustrating this process can be downloaded from http://www.lemos.uni-rostock.de/en/gallery/.

The paired vortices experience the azimuthal instability and takes the crude ring form. Later they are destroyed downstream in the region $1 < x/D < 6$. In far field at large $x/D$ the vortex structures look like a tree with branches oriented against the main flow direction (see Fig. 2.15).

![Figure 2.12: Development of instability in the jet (taken from [1]).](image)

Creation of vortex rings is the reason for the jet noise. The noise produced, for instance, by jet propulsors of airplanes is the action of these vortices. The vortices play a significant positive role in jet mixers widely used in food industry, chemical engineering, etc. That is why, one of the most per-
Figure 2.13: Development of instability in the free jet.

Figure 2.14: Development of instability in the free jet.

Figure 2.15: Vortex structures in a free jet in a far field.
spective way to reduce the noise or to increase mixing is the manipulation of vortices arising behind the jet nozzle. To increase the mixing it is necessary to strengthen the Kelvin Helmholtz vortices. To decrease the noise the vortices should be broken down into small ones. Fig. 2.16 shows the effect of the acoustic impact on jet. The original vortices (lower picture) are split into small ones (upper picture).

Figure 2.16: Vortex structures in a free jet with acoustic impact.

High resolution laser diagnostics methods LIF and PIV allow to get a deep inside into the structure of the turbulent flow. Fig. 2.17 shows the structure of the confined jet mixer flow displayed by Planar Laser Induced Fluorescence (PLIF) Method. The macrostructure obtained with low resolution is shown in the upper Figure. A small window with sizes $2.08\text{mm} \times 2.72\text{mm}$ was selected for high resolution PLIF measurements. The vortex microstructures are presented in the lower Figure. A very important observation is the presence of fine vortex structures. A gallery of such vortices obtained at different distances $x/D$, where $D$ is the diameter of the closing pipe, is given in Fig. 2.18. The smallest vortices are the so called Kolmogorov vortices which are considered in the next chapter.
2.3.3 Laminar- turbulent transition in wall bounded flows.

The boundary layer on a plate is the thin layer of rapid change of the velocity from zero to 99.5% of the incident flow. A possible scenario of the laminar turbulent transition in the boundary layer on a flat plate is shown in Fig. 2.19. First, the transversal vortices are generated in the boundary layer due to the so-called Tollmien-Schlichting instability. They experience the secondary instability and form downstream the lambda structures. The latter interact with each other and experience the tertiary instability. They lose their original regular form and become stochastic. An important feature of the turbulent boundary layer is the presence of streaks (strips) of the low velocity fluid regions (see Fig. 2.20). They arise due to induction of lambda structures.
2.3.4 Uneven distribution of the vorticity in the turbulent flows at large Reynolds number

In this subsection we analyze high resolved PIV measurements data obtained by Valery Zhadanov at the Institute of Technical Thermodynamics of the Rostock University. The flow is the turbulent axisymmetric jet developing in a coflow confined by a pipe of diameter $D = 50\, \text{mm}$ and length $5000\, \text{mm}$. A schematic of this flow system is given in Fig. 2.22. Medium in both flows is water. The inner tube had diameter $d = 10\, \text{mm}$ and the length $600\, \text{mm}$ chosen from the condition that perturbations caused by the knee bend are suppressed near the nozzle exit. The test section of the mixer was installed in a Perspex rectangular box filled with water to reduce refraction effects. More detailed information about the hydrodynamic channel can be found in [?]. Since the Reynolds number based on the jet exit velocity $U_d$ is $Re_d = dU_d/\nu = 10^4$ the jet can be considered as a fully-developed turbulent jet. PIV measurements were performed within the window $3.232\, \text{mm} \times 2.407\, \text{mm}$ with pixel distance of $\Delta = 68.8\, \mu\text{m}$. The laser thickness estimated as $\sim 40\, \mu\text{m}$ is very thin.

The measurement window was located on the centerline of the jet mixer at the distances $x/D = 1$ and 7 from the nozzle. The vorticity was calculated using the central differential scheme (CDS). The snapshots of the vorticity

Figure 2.18: Fine vortex structures in a confined jet mixer flow. PLIF measurements by Valery Zhdanov (LTT Rostock). Spatial resolution is $31\, \mu\text{m}$. Schematically shown in Fig. 2.21.
component squared $\omega_z^2 < \langle \omega_z^2 \rangle$, where $\langle \rangle$ stand for quantity averaged over
the window, is shown in Fig. 2.23. Strong uneven distribution of $\omega_z^2$ pointed
clearly out, that the vorticity is concentrated in a relatively small number
of spots or vortices. This character of the distribution is typical for both
initial development of the confined jet at $x/D = 1.0$ with weak anisotropy
($R_{11} = 0.09$, $R_{22} = 0.073$) and in the region of its strong decay at $x/D = 7.0$
where the flow is almost isotropic ($R_{11} = 0.036$, $R_{22} = 0.035$). Distribution
for the doubled two dimensional turbulent kinetic energy $u_x u_x + u_y u_y$, where $u_i$ are fluctuations, is shown in Fig. 2.24 for the same time instants.

Strong concentration of vorticity is especially obvious in Fig. 2.25. The
cells are sorted in order of descend of $\omega_z^2$, i.e. the first cell has the maximum
value of $\omega_z^2$ and the last one with the number $N$ has the minimum value.
The ratio $\varepsilon_k = \frac{\sum_{i=1}^{k} \omega_{zi}^2}{\sum_{i=1}^{N} \omega_{zi}^2}$ shows the contribution of $k$ cells to the total
amount $\Omega = \sum_{i=1}^{N} \omega_{zi}^2$. The ratio along the horizontal axis shows the fraction
of cells containing $\varepsilon_k$. As seen from this figure, the dependence $\varepsilon_k(k/N)$ is strongly nonlinear and reaches the saturation very quickly. Five percent of cells contains around fifty percent of the total $\Omega$, twenty percent of cells contains more than eighty percent, sixty percent of cells contains less than five percent of $\Omega$. Therefore, the number of active cells and, respectively, number of active vortices are very small. Note that the background vorticity obtained as the average over the whole measurement window is of order of $\sim 10^{-3}$ although the maximum and minimum values of a few dozens.

Figure 2.21: Conceptual model of the organization of the turbulence close to the wall proposed by Adrian et al. (2000).

Figure 2.22: Sketch of the flow. 1- knee bend of nozzle, 2- plate for damping of vortices shed from knee bend 1, 3- outer tube, 4- support plates, 5- nozzle, 6- test section, 7- water box.
Figure 2.23: Snapshot of the field $\omega_z^2/ < \omega_z^2 >$ within the measurement window in jet mixer. The averaged $< \omega_z^2 >$ was $1.19s^{-2}$ and $0.459s^{-2}$ at, respectively, $x/D = 1$ and 7.

Figure 2.24: Snapshot of the field $(u_x u_x + u_y u_y)/ < u_x u_x + u_y u_y >$ within the measurement window in jet mixer. The averaged $< u_x u_x + u_y u_y >$ was $0.011(m/s)^2$ and $0.0024(m/s)^2$ at, respectively, $x/D = 1$ and 7.
The distribution of \( E_k = \sum_{i=1}^{k} \frac{u_i^2}{\sum_{i=1}^{N} u_i^2} \) is less nonlinear indicating the fact that the distribution of the energy is more uniform than that of \( \omega^2 \). The field of the kinetic energy contains less spots as clearly seen from the comparison of Figures 2.23 and 2.24. The dependence \( E_k(k/N) \) is nonlinear only at the beginning and then becomes almost linear. The reason of more uniform distribution is that the energy is an integral quantity, whereas \( \omega_z \) is the local one. The contribution to \( E \) is carried out not only by vortices located at adjacent cells within the measurement plane but also by all vortices of the volume including vortices located outside of the measurement window. Uneven character of the \( \varepsilon_k(k/N) \) distribution, revealed above, is strengthened when the laser thickness of PIV measurements is decreased from \( \sim 400\mu m \) to \( \sim 40\mu m \) (Fig. 2.26).

The vortices are inclined to the measurement window at different angles \( \beta \). The trace of vortices on the measurement plane is \( \omega \sin \beta \). One can assume
that the maximum $\omega_z^2$ corresponds to vortices which are perpendicular to the measurement plane ($\beta = \pi/2$). Already visual analysis of Fig. 2.23 suggests that the strongest vortices are approximately axisymmetric. We apply the algorithm proposed in [?] to detect the vortex structures in the field of $\omega_z^2$ using two dimensional linear approximation of $\omega_z^2$. Note that the linear approximation is consistent with CDS applied for the calculation of $\omega_z$. Fig. 2.27 shows the probability density function of the structures of the field $\omega_z^2$. The most frequent structures have radius around $\sim 2.5\Delta$.

The p.d.f of the ratio $\frac{a-b}{\sqrt{ab}}$ indicating the circularity of the vortex cross section is given in Fig. 2.28. As seen from Fig. 2.28 the circular cross section corresponding to $a = b$ is the most frequent case. A large fraction of vortices is inclined to the measurement plane. Even if they are axisymmetric their intersection with measurement plane is not circular. It means that the true number of circular vortices is sufficiently larger than these corresponding to $\frac{a-b}{\sqrt{ab}} = 0$ in Fig. 2.28. Analysis of the circularity should be done with care because the peak of $\frac{a-b}{\sqrt{ab}} = 0$ can be just due to a low resolution. Indeed, if the real size of vortex is smaller than the cell size $\Delta$, being identified at any node of the grid, it occupies four adjacent cells. In our algorithm such a vortex is identified as the circle with the radius of $R = \Delta$. The circularity of vortices with the radius equal to $\Delta$ is indefinable. To exclude their influence we calculated conditioned p.d.f. of $\frac{a-b}{\sqrt{ab}}$ at $R > m\Delta$ shown in Fig. (2.29). As clearly seen the peak-like character of p.d.f. in vicinity of $\frac{a-b}{\sqrt{ab}} = 0$ is kept even at $m = 4$. Taking the fact into account, that the most frequent vortices have according to Fig. 2.27 $m \approx 2.5$, and results in Fig. 2.29, one can conclude that the axisymmetric approximation of fine vortices can be considered as quite appropriate. Increase of the order of spline approximation of $\omega_z^2$ field up to three doesn’t change the qualitative conclusions drawn from the bilinear approximation.

2.3.5 Distribution of the averaged velocity in the turbulent boundary layer

A remarkable feature of the turbulent boundary flow is the presence of three typical velocity distribution regions. The instantaneous velocity distributions can be quite different (s. Fig. 2.32). However, the averaged velocity has typical distribution close to the wall regardless of the flow type.

Let us introduce the following designations:

- $y$ is the distance from the wall,
- $\tau_w$ is the stress at the wall,
- $u_r = \sqrt{\frac{\tau_w}{\rho}}$ is the friction velocity,
• $y^+ = \frac{u^+ y}{\nu}$ is the dimensionless wall distance.

The stress in wall turbulence flow can be considered as the sum of the
Figure 2.30: Vertical distribution of the velocity $u_x$ at three different time instants in boundary layer.

Laminar and turbulent stresses:

$$\tau = \tau_l + \tau_t$$  \hspace{1cm} (2.12)

Close to the wall the turbulent fluctuations are weak. The laminar stress $\tau_l$ dominates over the turbulent one $\tau_t$, i.e. $\tau \sim \tau_l$. We consider the thin boundary layer, i.e. the stress is approximately equal to the wall stress $\tau_w$:

$$\tau \approx \tau_w$$  \hspace{1cm} (2.13)

Applying the Newton hypothesis (1.18) to the two dimensional wall bounded flow, one gets from (2.13)

$$\tau_w = \rho \nu \frac{du_x}{dy}$$  \hspace{1cm} (2.14)

or

$$\frac{u_x}{u_r} = y^+ + C$$  \hspace{1cm} (2.15)

From the condition at the wall $u_x = 0$ the unknown constant $C$ is zero, i.e.

$$\frac{u_x}{u_r} = y^+$$  \hspace{1cm} (2.16)

Close to the wall the velocity increases linearly $u_x \sim y$. This law confirmed in measurements is valid in the range $0 < y^+ < 5$. This region is referred to as the viscous sublayer.

Far from the wall the laminar stresses are smaller than the turbulent ones $\tau \sim \tau_l$. The turbulent stress $\tau_l$ can be found from the Prandtl mixing length
model. The instantaneous velocities are presented as the sum of averaged $\overline{u}$ and fluctuation $u'$ parts:

$$u_x = \overline{u}_x + u'_x, u_y = u'_y, \overline{u}_y = 0 \quad (2.17)$$

The averaged velocities are defined as

$$\overline{u}_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T u_x(t) dt \quad (2.18)$$

The turbulent stress $\tau_{12} = \tau_{xy}$ according to the definition is

$$\tau_{12} = \rho u'_x u'_y \quad (2.19)$$

Prandtl proposed in 1925 a very simple algebraic relation for $u'_x u'_y$ using ideas from the kinetic gas theory developed by Boltzmann. Let us consider the fluid particle at the distance $y$ from the wall. Let the particle velocity be equal to the averaged velocity at $y$: $\overline{u}_x$. Due to some perturbations the particle jumps from the position $y$ to the position $y + l_x$ and attains the fluid layer with the averaged velocity $\overline{u}_x + \frac{\partial u_x}{\partial y} l_x$. Since the particle has velocity $\overline{u}_x$, the velocity at the point $y + dy$ is changed. Obviously, this change is $-\frac{\partial u_x}{\partial y} l_x$, or:

$$\sqrt{u'_x^2} = \frac{\partial u_x}{\partial y} l_x \quad (2.20)$$
Root square of the averaged squared pulsation in vertical direction is written in a similar form:

\[ \sqrt{\langle u_y'^2 \rangle} = \frac{d\bar{\pi}_x}{dy} l_y \]  

(2.21)

Introducing the correlation coefficient \( R_{xy} = \frac{u'_x u'_y}{\sqrt{\langle u_x'^2 \rangle} \sqrt{\langle u_y'^2 \rangle}} \) and using (2.20) and (2.21) one gets:

\[ |\tau_{12}| = |\tau_t| = \rho R_{xy} l_x l_y \left( \frac{d\bar{\pi}_x}{dy} \right)^2 = \rho l^2 \left( \frac{d\bar{\pi}_x}{dy} \right)^2 \]  

(2.22)

where \( l^2 = R_{xy} l_x l_y \) is the mixing length of Prandtl. The mixing length is determined from empirical data. For instance, the length for the wall bounded flow is

\[ l = \kappa y \]  

(2.23)

\( \kappa \) is the first constant of the turbulence, or the constant of Karman. It is equal to 0.41. Van Driest proposed the modification of (2.23) to take the wall damping effect into account:

\[ l = \kappa y \left( 1 - e^{-\frac{y_+}{A}} \right) \]  

(2.24)

where \( A \) is the Van Driest constant, which is equal to 26 or 27. In shear flows \( l = \text{Const} \cdot \delta(x) \), where \( \delta \) is the shear layer thickness.

We consider again the thin boundary layer, i.e. the stress is approximately equal to the wall stress \( \tau_w = \tau_{12} \). Using (2.22) and (2.23) we get

\[ \tau_w = \rho l^2 \left( \frac{d\bar{\pi}_x}{dy} \right)^2 \]  

(2.25)

\[ \frac{d\bar{\pi}_x}{dy} = \frac{1}{l} \sqrt{\frac{\tau_w}{\rho}} = \frac{u_x}{u_r} \]  

(2.26)

The differential equation (2.26) reads

\[ \frac{u_x}{u_r} = \frac{1}{\kappa} \ln y^+ + C \]  

(2.27)

The constant \( C \) is approximately equal to 5.2. The region (2.27) is referred to as the logarithmic region which takes place within \( 30 < y^+ < 300 \). The region \( 5 < y^+ < 30 \) between the viscous and the logarithmic regions is called the buffer layer. The region at \( y^+ > 300 \) is the wake region. The results of the analysis are summarized in Fig. 2.22.
Figure 2.32: Structure of the velocity distribution in the turbulent boundary layer. \( U^+ = u_x/u_\tau \)
Chapter 3

Basic definitions of the statistical theory of turbulence

3.1 Reynolds averaging

Reynolds proposed to represent any stochastic quantity in turbulent flow as the sum of its averaged part and fluctuation. For instance, this representation applied for velocity components reads

\[
\begin{align*}
  u_x &= \bar{u}_x + u'_x; \\
  u_y &= \bar{u}_y + u'_y; \\
  u_z &= \bar{u}_z + u'_z; \\
\end{align*}
\] (3.1)

The Reynolds averaged velocities are

\[
\begin{align*}
  \bar{u}_x &= \lim_{T \to \infty} \frac{1}{T} \int_0^T u_x dt; \\
  \bar{u}_y &= \lim_{T \to \infty} \frac{1}{T} \int_0^T u_y dt; \\
  \bar{u}_z &= \lim_{T \to \infty} \frac{1}{T} \int_0^T u_z dt
\end{align*}
\] (3.2)

The averaged fluctuation is zero

\[
\bar{u}'_{x,y,z} = 0
\]

The root of the averaged square of fluctuations is called root mean square, or r.m.s.. The quantity averaged twice is equal to quantity averaged once \(\bar{u} = \bar{u}\).

If the turbulence process is statistically unsteady (for instance r.m.s is changed in time), the definition of the Reynolds averaging (3.2) is not applicable and should be extended using the concept of ensemble averaging.
Within the ensemble averaging the stochastic process is repeated N times from the initial state. The turbulent quantity $u$ is measured at a certain time $t^*$ N times. The ensemble averaged quantity is then:

$$\overline{u}(t^*) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u_i(t^*)$$

### 3.2 Isotropic and homogeneous turbulence

The turbulence is isotropic if r.m.s of all three velocity fluctuations are equal

$$u_x'^2 = u_y'^2 = u_z'^2$$  \hspace{1cm} (3.3)

The turbulence parameters are invariant with respect to the rotation of the reference system. The turbulence is homogeneous in some fluid volume if all statistical parameters are the same for all points in this volume, i.e. $u_{x,y,z}^2(x) = u_{x,y,z}^2(x + \vec{r})$. This equality can be written for all statistical moments. The turbulence parameters are invariant with respect to the translation of the reference system.

### 3.3 Correlation function. Integral length.

The product of two fluctuations is the correlation. The product of two fluctuations at two point separated by the distance $\vec{r}$ is the correlation function:

$$R_{ij}(\vec{x}, \vec{r}) = u_i'(\vec{x})u_j'(\vec{x} + \vec{r})$$  \hspace{1cm} (3.4)

If $i = j$ the correlation function $R_{ii}$ is referred as to the autocorrelation function. In homogeneous turbulence $R_{ii}$ depends only on the separation:

$$R_{ii}(\vec{r}) = u_i'(\vec{x})u_i'(\vec{x} + \vec{r})$$  \hspace{1cm} (3.5)

The coefficient of the autocorrelation function is changed between zero and one:

$$\rho_{ii}(\vec{x}, \vec{r}) = \frac{u_i'(\vec{x})u_i'(\vec{x} + \vec{r})}{u_i'^2(\vec{x})}$$  \hspace{1cm} (3.6)

A sample of the autocorrealtion function coefficient for scalar fluctuation $f'$

$$\rho_f(\vec{x}, \vec{r}) = \frac{f'(\vec{x})f'(\vec{x} + \vec{r})}{f'^2(\vec{x})}$$  \hspace{1cm} (3.7)
at three different points A, B and C across the jet mixer is shown in Fig. 3.1. In measurements presented in Fig. 3.1 the scalar $f$ is the concentration of the dye injected from the nozzle (see Fig. 3.1, low picture, right). The change of the function has a certain physical meaning. Let us consider the autocorrelation function with respect to the point C (blue line):

$$\rho_f(r_C, r) = \frac{f'(r_C)f'(r_C + r)}{f'^2(r_C)}$$

(3.8)

where $r$ is the radial coordinate across the pipe. The $\rho_f(r_C, r_A)$ is negative. It means the increase of the quantity $f$ at the point $C$ ($f'(r_C) > 0$) is followed by the decrease of this quantity at the point $A$ ($f'(r_A) < 0$). This is true in statistical sense, i.e. the most probable consequence of the increase $f(r_C)$ is the decrease of $f(r_A)$.

The correlation function and autocorrelation function can be written not only for spatial separation but also for separation in time. For example, the autocorrelation temporal function of the $u_i$ fluctuation is

$$\rho_{u_i}(\vec{x}, \tau) = \frac{u_i'(\vec{x}, t)u_i'(\vec{x}, t + \tau)}{u_i'^2(\vec{x}, t)}$$

(3.9)
The integral of the spatial autocorrelation functions

\[ L_{ij}(\vec{x}) = \int_0^\infty \rho_{ii}(\vec{x}, x_j) dx_j \quad (3.10) \]

is the integral length. The integral lengths are estimations of the size of the largest vortex in flow. A sample of the integral length of the scalar field \( f \) along the jet mixer centerline (Fig. 3.1, right)

\[ L_f(x) = \int_{-D/2}^{D/2} \rho_f(r) dr \quad (3.11) \]

is shown in fig. 3.2, where \( \rho_f \) is the autocorrelation function across the jet mixer, \( d \) is the nozzle diameter, \( D \) is the diameter of the closing pipe. \( L_f \) is the estimation of the largest structure of the scalar field (in this case, the size of the spot of colored liquid injected from the nozzle). The integral of the temporal autocorrelation functions

\[ T_i(\vec{x}) = \int_0^\infty \rho_{ii}(\vec{x}, \tau) d\tau \quad (3.12) \]

is the integral time length. Coefficients of the autocorrelation function of the axial velocity fluctuations for the free jet are presented in Fig. 3.3. The line 1 corresponds to the autocorrelation function calculated along the jet boundary.
Figure 3.3: Autocorrelation functions in free jet flow.

line (shown by the blue line in Fig. 3.3, right). The line 2 corresponds to the autocorrelation function calculated along the jet axis. Oscillating character of the dependency \( R_{uu}(\Delta x/d) \) indicates the presence of vortex structures arising at the jet boundary and attaining the jet axis. The distance between zero points is roughly the vortex size.

3.3.1 Some relations in isotropic turbulence

In the isotropic turbulence \( u'^2 = u'_i(x)u'_i(x) = u'_l(x)u'_l(x) = ... \) the autocorrelation function can be represented in the form [5]

\[
R_{ij} = u^2 \left( f - \frac{g}{r^2} r_i r_j + g \delta_{ij} \right) \tag{3.13}
\]

where

\[
f(r) = \frac{u'_i(x)u'_i(x+r)}{u'_l(x)u'_l(x)} \tag{3.14}
\]

is the autocorrelation of the longitudinal velocity calculated in longitudinal direction. For instance, the autocorrelation function of the \( u_x \) fluctuation calculated in \( x \) direction,

\[
f(r) = \frac{u'_x(x)u'_x(x+r)}{u'_x(x)u'_x(x)} \tag{3.15}
\]

or the autocorrelation function of the \( u_y \) fluctuation calculated in \( y \) direction:

\[
f(r) = \frac{u'_y(y)u'_y(y+r)}{u'_y(y)u'_y(y)} \tag{3.16}
\]

53
The function $f(r)$ is the same in both cases (3.15) and (3.16). The autocorrelation function is calculated for transversal velocities along any direction

$$g(r) = \frac{u'_t(x)u'_t(x + r)}{u'_t(x)u'_t(x)}$$  \hspace{1cm} (3.17)

For instance, the autocorrelation function of the $u_x$ fluctuation calculated in $y$ direction,

$$g(r) = \frac{u'_x(y)u'_x(y + r)}{u'_x(y)u'_x(y)}$$  \hspace{1cm} (3.18)

or the autocorrelation function of the $u_y$ fluctuation calculated in $x$ direction:

$$g(r) = \frac{u'_y(x)u'_y(x + r)}{u'_y(x)u'_y(x)}$$  \hspace{1cm} (3.19)

The function $g(r)$ is the same in both cases (3.18) and (3.19). The velocities components used in the previous definitions are illustrated in Fig. 3.4. $u_t$ is the velocity pulsation vector, $u_{il}$ is its projection on the direction connecting two points (i.e. longitudinal direction), $u_{it}$ is its projection on the transversal direction. Products like $u_{il}u_{it}$ are the correlations between points 1 and 2.

The following relations are valid between $g$ and $f$ in the isotropic homogeneous turbulence (see [5]):

$$g = f + \frac{1}{2} r \frac{\partial f}{\partial r}$$  \hspace{1cm} (3.20)

Typical form of $f$ and $g$ is shown in Fig. 3.5. The change of the sign of $g$ function is due to the continuity equation of the velocity field. The integral length calculated using $f$ is twice as large as that calculated using $g$. 

54
3.3.2 Taylor microscale \( \lambda \)

Until Kolmogorov derived his estimations for vortices in 1941, it has been thought that the minimum vortices arising in the turbulent flow have sizes estimated by Taylor. Let us consider the parabola fitted to the autocorrelation function at point \( r = 0 \). The parabola intersects the horizontal axis at a certain point. The coordinate of this point \( \lambda \) is the scale introduced by Taylor and called as the Taylor microscale. The Taylor microscale can be calculated through the second derivative of the autocorrelation function at \( r = 0 \). The Taylor series of \( f \) in the vicinity of the point \( r = 0 \) is

\[
f(r) = 1 + \frac{1}{2} \frac{\partial^2 f}{\partial r^2}(0) r^2 + O(r^4) \quad (3.21)
\]

The parabola fitted to the curve \( f(r) \) at \( r = 0 \) intersects the horizontal axis at the point:

\[
\lambda_f = \sqrt{-\frac{2}{\frac{\partial^2 f}{\partial r^2}(0)}} \quad (3.22)
\]

Similar relations can be derived for the transversal autocorrelation

\[
\lambda_g = \sqrt{-\frac{2}{\frac{\partial^2 g}{\partial r^2}(0)}} \quad (3.23)
\]

Today the Taylor microscale \( \lambda \) is still in use in turbulent research although it has no physical meaning. Very popular is the Reynolds number based on the Taylor microscale

\[
Re = u'\lambda/\nu \quad (3.24)
\]

which characterizes the state of the turbulence in the flow.
3.3.3 Correlation functions in the Fourier space

Any continuous function can be represented in the Fourier space:

\[ f(\vec{r}, t) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \hat{f}(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}} d\vec{k} \]  \hspace{1cm} (3.25)

The new function \( \hat{f}(\vec{k}, t) \) is then known as the Fourier transform and/or the frequency spectrum of the function \( f \). The Fourier transform is also a reversible operation:

\[ \hat{f}(\vec{k}, t) = \int_{-\infty}^{\infty} f(\vec{r}, t) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \]  \hspace{1cm} (3.26)

The Fourier transformation can be also written for the correlation function:

\[ R_{ij}(\vec{r}) = \int_{-\infty}^{\infty} \Phi_{ij}(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d\vec{k}, \quad \Phi_{ij}(\vec{k}) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} R_{ij}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \]  \hspace{1cm} (3.27)

Very often one uses one dimensional correlation functions defined as

\[ \Theta_{ij}(k_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ij}(r_1, 0, 0) e^{-ik_1r_1} dr_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(k_1, k_2, k_3) dk_2 dk_3 \]  \hspace{1cm} (3.28)

**Proof of the formula (3.28)** The inverse Fourier transform of the function \( R_{ij}(r_1, 0, 0) \):

\[ \Theta_{ij}(k_1, 0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ij}(r_1, 0, 0) e^{ik_1r_1} dr_1 \]  \hspace{1cm} (3.29)

The inverse transformation reads:

\[ R_{ij}(r_1, 0, 0) = \int_{-\infty}^{\infty} \Theta_{ij}(k_1, 0, 0) e^{-ik_1r_1} dr_1 \]  \hspace{1cm} (3.30)

The general definition is

\[ R_{ij}(r_1, r_2, r_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(k_1, k_2, k_3) e^{i\vec{k}\cdot\vec{r}} d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \]  \hspace{1cm} (3.31)
From the last formula we have:

\[ R_{ij}(r_1, 0, 0) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(k_1, k_2, k_3) dk_2 dk_3 \right) e^{ik_1 r_1} dk_1 \]  

(3.32)

Comparison of (3.32) with (3.30) results in the desired formula

\[ \Theta_{ij}(k_1, 0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}(k_1, k_2, k_3) dk_2 dk_3 \]  

(3.33)

### 3.3.4 Spectral density of the kinetic energy

According to the definition of the correlation function

\[ R_{ij}(\vec{r}) = u_i(\vec{x})u_j(\vec{x} + \vec{r}) \]  

(3.34)

the total kinetic energy is

\[ TKE = \frac{1}{2} R_{ii}(0) = \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{ii}(\vec{k}) d\vec{k} = \int_{0}^{\infty} \Phi_{ii}(\vec{k}) d\vec{k} \]  

(3.35)

The quantity

\[ E(k) = \int_{|\vec{k}|} \Phi_{ii}(\vec{k}) d\vec{k} \]  

(3.36)

is the spectral density of the kinetic energy. Physically it is the energy on the sphere \( k = \sqrt{k_1^2 + k_2^2 + k_3^2} \) in the Fourier space. The total kinetic energy is then:

\[ TKE = \int_{0}^{\infty} E(k) dk \]  

(3.37)

### 3.4 Structure functions

#### 3.4.1 Probability density function

We take the definitions from Wikipedia: In probability theory, a probability density function (pdf), or density of a continuous random variable, is a function that describes the relative likelihood for this random variable to take
on a given value. The probability for the random variable to fall within a particular region is given by the integral of this variables density over the region. The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one.

### 3.4.2 Structure function

Kolmogorov introduced the structure function of $q$ order for any stochastic function. For instance, for the longitudinal velocity along the longitudinal direction (see Fig. 3.5) it reads:

$$S_q(l) = \langle (u_{2l} - u_{1l})^q \rangle$$  \hspace{1cm} (3.38)

The standard deviation squared is then $\sigma^2 = S_2$. If the p.d.f. of the structure function is described by the Gaussian function

$$p.d.f.(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$  \hspace{1cm} (3.39)

where $\mu$ is the mean value of the stochastic value, the turbulence is Gaussian. In reality, the most of the turbulence parameters are not Gaussian. The deviations from the Gaussian turbulence is characterized by the Kurtosis $Kurt$ and Skewness $Sk$. The Kurtosis

$$Kurt = \frac{\langle (u_{2l} - u_{1l})^4 \rangle}{\langle (u_{2l} - u_{1l})^2 \rangle^2}$$  \hspace{1cm} (3.40)

is three for the Gaussian turbulence. Big values of the Kurtosis means that the p.d.f. distribution of the structure function $S_1(l) = \langle u_{2l} - u_{1l} \rangle$ is very flat. The Kurtosis is also often called as flatness. If Kurtosis for small $l \sim 0$ is big, it means that the field of the stochastic field is very intermittent. Big differences are possible even the separation between two points $l$ is small.

The p.d.f. function has long tails in this case. A sample of Kurtosis for the scalar structure function $S(x) = f(x + r) - f(x)$ is given in Fig.3.6. The skewness

$$Sk = \frac{\langle (u_{2l} - u_{1l})^3 \rangle}{\langle (u_{2l} - u_{1l})^2 \rangle^{3/2}}$$  \hspace{1cm} (3.41)

is zero for the Gaussian process. For the isotropic turbulence the Skewness of the derivative

$$Sk = \frac{\langle \left( \frac{\partial u_l}{\partial t} \right)^3 \rangle}{\langle \left( \frac{\partial u_l}{\partial t} \right)^2 \rangle^{3/2}}$$  \hspace{1cm} (3.42)
Figure 3.6: Kurtosis of the structure function for the concentration of the scalar field obtained in the jet mixer.

is $-0.5$. Physically it means that negative values of the derivative $\frac{\partial u_i}{\partial l}$ are more probable than positive ones. Please prove that the skewness (3.42) is the skewness of the structure functions of the first order $S_1(l) = \langle u_{2l} - u_{1l} \rangle$ calculated at $l \to 0$. 
4.1 Physical background

One of the most outstanding results in turbulence theory was obtained by Kolmogorov in 1941. The Kolmogorov theory known as K41 is based on the hypothesis of local isotropy of the turbulent motion at small scales. The physical model behind the Kolmogorov theory is the vortex cascade illustrated in Fig. 4.2. Big vortices with scales $L$ (corresponds to the wave numbers $\pi/L$ in the Fourier space) break up to small ones, which in turn split into even smaller and so on up to the smallest vortices with the scale $\eta$. One of the most important vortex break up mechanisms is the vortex reconnection described above. The energy is transferred from big vortices to small ones almost without the loss. The massive dissipation $\varepsilon$ takes place at small vortices referred to as the dissipative or the Kolmogorov vortices. The real turbulent vortices are similar to those calculated by Isazawa et al. (Fig. 4.3). Vortices are displayed at three different time instants. The upper pictures are obtained from the lower ones by filtering out the high frequencies. As seen big vortices are revealed in low frequency simulation. If the resolution is increased more and more small scale vortex filaments appears on the place of big smooth vortices. Thus, the most important physical processes during the vortex break up are:

- Transfer energy from large scales to small ones and

- Dissipation of the energy in small vortices.
Figure 4.1: Andrey Kolmogorov was a mathematician, preeminent in the 20th century, who advanced various scientific fields (among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity).

Figure 4.2: Illustration of the vortex cascado
4.2 Dissipation rate

Two parameters, which are of importance during the cascado process, are the kinematic viscosity \( \nu \) and the dissipation rate \( \varepsilon \). The energy dissipation rate per unit mass of a turbulent fluid is given by

\[
\varepsilon = \frac{\nu}{2} \sum_{ij} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)^2 = 2\nu s_{ij} s_{ij} \tag{4.1}
\]

where

\[
s_{ij} = \frac{1}{2} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) \tag{4.2}
\]

The dissipation \( \varepsilon \) is a random function of the coordinates and time, which fluctuates together with the field \( \mathbf{u}(x,t) \). In what follows we consider the mean dissipation rate \( \bar{\varepsilon} \) designating it as \( \bar{\varepsilon} \). The energy dissipated by the small vortices is generated by large scale vortices. The energy production is defined as

\[
P = \bar{u}_i' u_j' \frac{\partial \bar{u}_i}{\partial x_j} \tag{4.3}
\]

Based on the dimension analysis, Prandtl and Kolmogorov proposed the estimation of the integral length of the turbulent flow.
The formula (4.4) is valid for very high Reynolds numbers for the turbulence being in the equilibrium, i.e. the production of the turbulence is compensated by its dissipation, i.e. $P = \varepsilon$.

4.3 Kolmogorov hypotheses

The basis of the Kolmogorov theory are three hypotheses, which are supposed to be valid for high Reynolds numbers $Re_t = \frac{UL}{\nu}$, where $v = \sqrt{TKE} = \sqrt{k}$ is the characteristic fluctuation velocity.

The Kolmogorov hypothesis of local isotropy reads:

At sufficiently high Reynolds number $Re_t$, the small-scale turbulent motions ($l \ll l_{EI}$) are statistically isotropic

Here $l_{EI}$ is the lengthscale as the demarcation between the anisotropic large eddies and the isotropic small eddies. Kolmogorov argued that all information about the geometry of the large eddies - determined by the mean flow field and boundary conditions - is also lost. Directional information at small scales is lost. With the other words, the direction of the vorticity vector $\omega$ of small turbulent vortices is uniformly distributed over the sphere. As a consequence, the statistics of the small-scale motions are in a sense universal - similar in every high-Reynolds-number turbulent flow (see [6]).

The Kolmogorov first similarity hypothesis reads:

In every turbulent flow at sufficiently high Reynolds number $Re_t$, the statistics of the small-scale motions ($l < l_{EI}$) have a universal form that is uniquely determined by $\nu$ and $\varepsilon$.

For this range of scales we can introduce characteristic size $\eta$, characteristic velocity $u_\eta$ and characteristic time $\tau_\eta$ which depend only on two parameters $\nu$ and $\varepsilon$:

$$\eta = \nu^{\alpha_\eta} \varepsilon^{\beta_\nu}, \quad u_\eta = \nu^{\alpha_u} \varepsilon^{\beta_u}, \quad \tau_\eta = \nu^{\alpha_\tau} \varepsilon^{\beta_\tau}$$  \hspace{1cm} (4.5)

The analysis of dimension allows one to derive the following dependences:
\[ \eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4}, \]
\[ u_\eta = (\nu \varepsilon)^{1/4}, \]
\[ \tau_\eta = (\nu/\varepsilon)^{1/2} \]
\[ (4.6) \]

Here \( \eta \) is the scale of the smallest dissipative vortices (Kolmogorov scale), \( u_\eta \) the characteristic velocity of turning of Kolmogorov vortices, \( \tau_\eta \) characteristic turn over time of Kolmogorov vortices.

Using expressions
\[ \varepsilon \approx \frac{k^{3/2}}{L} \quad u \approx k^{1/2} \]
\[ (4.7) \]

some useful estimations can be derived from (4.6):

\[ \eta/L \approx (Re_t)^{-3/4}, \]
\[ u_\eta/u = (Re_t)^{-1/4}, \]
\[ \tau_\eta/T = (Re_t)^{-1/2} \]
\[ (4.8) \]

Very remarkable is the first formula defining the ratio between the smallest and largest vortices in the flow. If \( L \) is, say one meter, and the fluctuation \( 1m/s \), the turbulent Reynolds number in water is \( Re_t = 10^6 \). The Kolmogorov scale is in this case 32000 as less as the flow macroscale \( L \). Estimations of the Kolmogorov scale in the jet mixer with nozzle diameter of \( d = 1cm \) and closing pipe of \( D = 5cm \) diameter is shown in Fig. 4.4.

The **Kolmogorov second similarity hypothesis** reads:

*In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale \( l \) in the range \( L \gg l \gg \eta \) have a universal form that is uniquely determined by \( \varepsilon \), independent of \( \nu \).*

This range is called as the inertial subrange. Since the vortices of this range are much larger than Kolmogorov vortices, we can assume that their Reynolds numbers \( lu_\eta/\nu \) are large and their motion is little affected by the viscosity. The energy density depends on the wave number \( k \) and the dissipation rate \( \varepsilon \)

\[ E(k) = \varepsilon^\alpha k^\beta \]
\[ (4.9) \]
The analysis of dimension leads to the Kolmogorov law

$$E(k) = \alpha \varepsilon^{2/3} k^{-5/3}$$

where $\alpha \approx 1.5$ is the constant.

### 4.4 Three different scale ranges of turbulent flow

Three different ranges can be distinguished in the spectrum of scales in the full developed turbulence at high Reynolds numbers $Re_l$ (Fig. 4.5):

- Energy containing range at $l > l_{EI}$ (according to Pope [6], $l_{EI} \approx \frac{L}{\nu} L$). Within this range the kinetic energy of turbulence is generated and big turbulent eddies are created.

- Inertial subrange at $l_{DI} > l > l_{EI}$ (according to Pope [6], $l_{DI} \approx 60 \eta$). Within this subrange the energy is transferred along the scales towards dissipative vortices without any significant loss, i.e. $\varepsilon \sim 0$. The energy density obeys the Kolmogorov law (4.10).

- Dissipation range $l < l_{DI}$. The dissipation of the energy of big vortices occurs within the dissipation range.
The inertial subrange and dissipation range belong to the universal equilibrium range. Three corresponding ranges can be distinguished in the distribution of the energy density over the wave numbers $k$. Fig. 4.6. The presence of the inertial and dissipation subranges was confirmed in numerous experimental measurements performed after development of the K41 theory (see Fig. 4.7, 4.8).

Figure 4.5: Three typical scale ranges in the turbulent flow at high Reynolds numbers

Figure 4.6: Three typical ranges of the energy density spectrum in the turbulent flow at high Reynolds number. 1- energy containing range, 2- inertial subrange, 3- dissipation range.
Figure 4.7: Experimental confirmation of the Kolmogorov law. The compensated energy spectrum for different flows.
4.5 Classification of methods for calculation of turbulent flows.

The energy spectrum Fig. 4.6 is used to classify three main methods of turbulent flows modelling (Fig. 4.9). The most general strategy is the Direct Numerical Simulation (DNS). Within the DNS the whole spectrum of turbulent structures is modelled starting from the biggest vortices of the energy containing range till the smallest dissipative Kolmogorov vortices. The Large Eddy Simulation (LES) models the energy containing vortices and a fraction of vortices corresponding to the inertial subrange. The effect of remaining vortices is considered using different approximation models. Since small vortices are universal, the models are also supposed to be universal. The Reynolds averaged Navier Stokes (RANS) models are dealing with the large vortices corresponding to the energy containing range. The effect of other vortices is taken by different semi-empiric models which are not universal.

4.6 Limitation of K-41. Kolmogorov theory K-62

The strongest and simultaneously the most questionable assumption of the Kolmogorov-41 is: Dissipation rate is an universal constant for each turbulent flow. Already in 1942, during a scientific seminar the Nobel price laureate
Landau noted, that the dissipation rate is a stochastic function, it is not constant. We consider the consequences of the neglect of this fact. According to the Kolmogorov-Obukhov law the structure function of the $q$-th order

$$S_q(l) = \langle (u_2 - u_1)^q \rangle$$  \hspace{1cm} (4.11)

has the following asymptotic behaviour at small $l$

$$S_q(l) \sim (\varepsilon l)^q \sim (l)^{\zeta q}$$  \hspace{1cm} (4.12)

Fig. 4.11 shows that the predictions of Kolmogorov and Obukhov deviate from measurement data. The reason of the discrepancy is the physical phe-
nomenon called the intermittency. The intermittency is caused by the presence of laminar spots in every turbulent flows even at very high Reynolds numbers.

Figure 4.11: Power of the structure function. Experiments versus prediction of Kolmogorov and Obukhov

After the deviation between the K-41 and measurement was documented, Kolmogorov tried to improve his theory. New Kolmogorov theory called as K-62 was published in 1962.

New theory is based on two following assumptions:

- Assumption 1:
  \[ S_q(l) = \langle \delta v_q^q \rangle \sim \langle \varepsilon_l^{q/3} \rangle \sim \frac{q}{l^{q/3}}, \]
  \[ \zeta_q = \frac{q}{3} + \tau_{q/3} \quad <\varepsilon_l^q > \sim l^{q/3} \quad (4.13) \]

- Assumption 2:
  \[ P(\varepsilon_l) = ce^{a \ln \varepsilon - a} \quad a = \ln \bar{\varepsilon} \quad \sigma^2_l = A + \mu \ln (L/l) \]
  \[ \tau_q = \frac{\mu}{2} q(1 - q) \quad \zeta_q = \frac{q}{3} + \frac{\mu}{18} q(3 - q) \quad <\varepsilon_l^2 > \sim l^{-\mu} \quad (4.14) \]

Unfortunately, various experiments showed later that the second assumption is proved to be wrong.
Chapter 5

Reynolds Averaged Navier Stokes Equation (RANS)

According to the Reynolds averaging each fluctuating quantity is represented as the sum of the averaged value and its fluctuation:

\[ u_x = \bar{u}_x + u'_x; \quad u_y = \bar{u}_y + u'_y; \quad u_z = \bar{u}_z + u'_z \]  \hspace{1cm} (5.1)

where the averaged part is defined as:

\[ \bar{u}_x = \frac{1}{T} \int_0^T u_x dt; \quad \bar{u}_y = \frac{1}{T} \int_0^T u_y dt; \quad \bar{u}_z = \frac{1}{T} \int_0^T u_z dt \]  \hspace{1cm} (5.2)

The Reynolds averaging has the following properties:

- averaged fluctuation is zero:
  \[ \bar{f}' = 0 \]  \hspace{1cm} (5.3)

- double averaged quantity is equal to once averaged one:
  \[ \bar{\bar{f}} = \bar{f} \]  \hspace{1cm} (5.4)

- averaged sum is equal to the sum of averaged:
  \[ \bar{f} + \bar{g} = \bar{f} + \bar{g} \]  \hspace{1cm} (5.5)

- operators of averaging and differentiation commutate:
  \[ \frac{\partial \bar{f}}{\partial t} = \frac{\partial \bar{f}}{\partial t}; \quad \frac{\partial \bar{f}}{\partial x} = \frac{\partial \bar{f}}{\partial x} \]  \hspace{1cm} (5.6)
• averaged product of two fluctuating quantities is not zero:
  \[ \overline{f'g'} \neq 0, \quad \overline{fg} = \overline{f} \overline{g} \quad (5.7) \]

• averaged product of any averaged quantity and fluctuation is zero:
  \[ \overline{fg'} = \overline{f} \overline{g'} = 0 \quad (5.8) \]

The starting point of the derivation of the RANS equation is the original Navier Stokes (NS) equation:

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = F_i + \frac{1}{\rho} \frac{\partial \tau_{ji}}{\partial x_j} \quad (5.9) \]

Here we use the summation convention of Einstein:

\[ u_j \frac{\partial u_i}{\partial x_j} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} \quad (5.10) \]

The NS equation is supplied with the continuity equation, which for the case of incompressible flow takes the form:

\[ \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \quad (5.11) \]

Using the continuity equation the convective term is written in the conservative form:

\[ u_j \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j} = \frac{\partial (u_i u_j)}{\partial x_j} \quad (5.12) \]

With (5.12) the NS equation reads

\[ \frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = F_i + \frac{1}{\rho} \frac{\partial \tau_{ji}}{\partial x_j} \quad (5.13) \]

This equation is valid for fluctuating quantities represented in Reynolds form (5.1)

\[ \frac{\partial (\bar{u}_i + u'_i)}{\partial t} + \frac{\partial (\bar{u}_i + u'_i)(\bar{u}_j + u'_j)}{\partial x_j} = \bar{F}_i + F' + \frac{1}{\rho} \frac{\partial (\bar{\tau}_{ji} + \tau'_{ji})}{\partial x_j} \quad (5.14) \]

Both r.h.s. and l.h.s. are averaged:

\[ \frac{\partial (\bar{u}_i + u'_i)}{\partial t} + \frac{\partial (\bar{u}_i + u'_i)(\bar{u}_j + u'_j)}{\partial x_j} = \bar{F}_i + F' + \frac{1}{\rho} \frac{\partial (\bar{\tau}_{ji} + \tau'_{ji})}{\partial x_j} \quad (5.15) \]
Utilization of Reynolds averaging properties results in:

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_i \bar{u}_j + \bar{u}'_i \bar{u}'_j)}{\partial x_j} = \bar{F}_i + \frac{1}{\rho} \frac{\partial \bar{\tau}_{ji}}{\partial x_j}$$  \quad (5.16)$$

Writing the term $\bar{u}'_i \bar{u}'_j$ on the r.h.s we obtain the Reynolds averaged Navier Stokes equation (RANS). Its unsteady version is called as the unsteady Reynolds averaged NS equation (URANS):

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = \rho \bar{F}_i + \frac{\partial}{\partial x_j} (\bar{\tau}_{ji} - \rho \bar{u}'_i \bar{u}'_j)$$  \quad (5.17)$$

There are two important features of URANS in comparison with NS:

- The URANS is written for the averaged quantities, whereas the NS for instantaneous ones,
- The URANS has additional term on the r.h.s. $-\rho \bar{u}'_i \bar{u}'_j$ which is called the Reynolds stress $R_{ij}$.

Generally the Reynolds stress is the matrix with nine terms:

$$R_{ij} = \begin{bmatrix} -\rho \bar{u}'_x \bar{u}'_x & -\rho \bar{u}'_x \bar{u}'_y & -\rho \bar{u}'_x \bar{u}'_z \\ -\rho \bar{u}'_y \bar{u}'_x & -\rho \bar{u}'_y \bar{u}'_y & -\rho \bar{u}'_y \bar{u}'_z \\ -\rho \bar{u}'_z \bar{u}'_x & -\rho \bar{u}'_z \bar{u}'_y & -\rho \bar{u}'_z \bar{u}'_z \end{bmatrix}$$  \quad (5.18)$$

Due to symmetry conditions the number of unknown stresses is six. The term $-\rho \bar{u}'_i \bar{u}'_j$ is called the stress since it has the same appearance in NS equation as the laminar stress:

$$\tau_{ji} = \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - p \delta_{ij}$$  \quad (5.19)$$

Laminar stress appears due to viscosity effects whereas the Reynolds stress is caused by flow fluctuations. Now the system of four fluid equations (three URANS+ continuity) has ten unknowns: three averaged velocity components $\bar{u}_i$, averaged pressure $\bar{p}$ and six Reynolds stresses. The system of fluid dynamics is not closed. Additional relations are necessary to express the Reynolds stresses through the velocities and pressure. This problem of determination of Reynolds stresses is called as the closure problem of the turbulence. A huge amounts of closure models was developed within the framework of URANS methodology. As shown above the URANS methods models the biggest vortices of the flow. The resting scales, which are filtered by the Reynolds avergaing out, are big enough and not universal. This is
the reason why the URANS models are not universal. Most of them are of semi empirical character. They are based on submodels with few constants selected for simple canonical flows. Non universality of closure models is the biggest weakness of URANS turbulence modelling.

The majority of URANS models used in engineering are based on the Boussinesq hypothesis which is the formal extension of the Newton hypothesis to turbulent flows. Boussinesq proposed to express the Reynolds stress through the strain rate tensor

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j} \right) \]

in the form of the Newton hypothesis with the only difference that instead of the kinematic viscosity \( \nu \) the turbulent viscosity \( \nu_t \) is used

\[-\rho u'_i u'_j = \rho \nu_t \left( \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j} \right) - \frac{2}{3} \rho \delta_{ij} k \quad (5.20)\]

In the simplest form for flow along the plate with \( \bar{u}_x(y) \) and \( \bar{u}_y = \bar{u}_z = 0 \) the formula (5.20) reads

\[-\rho u'_x u'_y = \rho \nu_t \frac{d\bar{u}_x}{dy} \quad (5.21)\]

The last term in (5.20) is introduced to keep the consistency. Indeed the sum of three diagonal terms of the Reynolds matrix is equal to the turbulent kinetic energy \( k = \frac{1}{2} \bar{u}'_i \bar{u}'_i \). Without this term the sum of r.h.s of (5.20) would result in the sum of the diagonal terms of the strain rate matrix \( S_{11} + S_{22} + S_{33} \) which is zero due to the continuity equation. It would be wrong result because \( k \neq 0 \). The turbulent closures (5.20) are referred to as the isotropic because the coefficient \( \nu_t \) is equal for all matrix elements \( R_{ij} \). While the kinematic viscosity depends on the liquid, the turbulent kinematic viscosity depends on the turbulent state of the flow. According to estimation of Landau the ratio of the kinematic viscosity to the turbulent one is proportional to the ratio of the Reynolds number to that corresponding to the transition for this type of flow

\[ \nu_t/\nu \sim Re/Re_{crit} \quad (5.22) \]

The URANS closure models are subdivided into algebraic and differential ones. The most prominent model amount the algebraic models is the Prandtl model described above. The disadvantages of the algebraic models are:

- they are good only for the simplest flow,
- not suitable for 3D flows,
• not suitable for separation flows,
• turbulent viscosity depends on averaged values of velocities,
• do not consider the flow history.

These disadvantages can be overcome using differential models which are subdivided into one, two and multi equation models. One and two equation models are usually isotropic based on the Boussinesq approach (5.20). Among the one equation models the most modern and efficient is the model of Spalart Allmares (SA model) written for the modified kinematic turbulent viscosity $\nu$. The equation for $\nu$ reads

$$\frac{\partial \nu}{\partial t} + \bar{u}_j \frac{\partial \nu}{\partial x_j} = C_{b1} \xi \nu - C_{w1} f_w \left( \frac{\nu}{d} \right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial x_k} \left( (\nu + \nu) \frac{\partial \nu}{\partial x_k} \right) + \frac{C_{b2}}{\sigma} \frac{d \nu}{d x_k} \frac{d \nu}{d x_k}$$

where

$$C_{b1} = 0, 1355, \quad C_{b2} = 0, 622, \quad C_{w1} = 7, 1, \quad \sigma = 2/3,$n
$$C_{w2} = 0, 3, \quad C_{w3} = 2, 0, \quad k = 0, 41 \quad (5.24)$$

$$f_{\nu1} = \frac{\chi^3}{\chi^3 + C_{\nu1}^3}, \quad f_{\nu2} = \frac{\chi}{1 + \chi f_{\nu1}}, \quad f_w = g \left[ \frac{1 + C_{w3}^6}{g^6 + C_{w3}^6} \right]^{1/6},$$

$$\chi = \nu \nu, \quad g = r + C_{w2}(r^6 - r), \quad r = \frac{\nu}{\xi k^2 d^2} \quad (5.25)$$

$$\xi = S + \frac{\nu}{k^2 d^2} f_{\nu2}, \quad S = \sqrt{2 \Omega_{ij} \Omega_{ij}}, \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

Within the more advanced $k - \varepsilon$ model the turbulent kinetic energy and the dissipation rate are calculated from the transport equations:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \left( \nu + \nu \frac{\tau_{ij}}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ (\nu + \nu \frac{\tau_{ij}}{\sigma_{\varepsilon}}) \frac{\partial \varepsilon}{\partial x_j} \right] + \frac{C_{\varepsilon1} \varepsilon}{k} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{C_{\varepsilon2} \varepsilon^2}{k}$$

If $k$ and $\varepsilon$ are known the turbulent viscosity $\nu_t$ can be found from the dimension analysis, applied to the dissipation rate.
\[ \varepsilon \approx k^{3/2}/L \]  \hspace{1cm} (5.28)

turbulent kinematic viscosity

\[ \nu_t = C_\mu \sqrt{kL} \]  \hspace{1cm} (5.29)

Here \( C_\mu = 0.09 \) is the empirical constant. Substitution of (5.28) into (5.29) leads to the sought relation:

\[ \nu_t = C_\mu \sqrt{kL} = C_\mu \frac{k^2}{\varepsilon} \]  \hspace{1cm} (5.30)

If \( \nu_t \) is known, the Reynolds stresses can be determined from the Boussinesq approach (5.20).
Chapter 6

Reynolds Stress Model (RSM)

6.1 Derivation of the RSM Equations

6.1.1 Step 1

The \( k \)-th Navier-Stokes equation

\[
\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j} (u_j u_k) = -\frac{1}{\rho} \frac{\partial p}{\partial x_k} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \tau_{jk} \tag{6.1}
\]

is multiplied with the velocity component \( u_i \)

\[
u_i \left( \frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j} (u_j u_k) \right) = u_i \left( -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \tau_{jk} \right) \tag{6.2}
\]

The \( i \)-th equation is multiplied with \( k \)-th velocity component:

\[
u_k \left( \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j u_i) \right) = u_k \left( -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \tau_{jk} \right) \tag{6.3}
\]

Resulting equations (6.2) and (6.3) are then summed:

\[
\frac{\partial (\rho u_i u_k)}{\partial t} + \frac{\partial (\rho u_i u_k u_j)}{\partial x_j} = -u_i \frac{\partial p}{\partial x_k} - u_k \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{jk}}{\partial x_j} + u_k \frac{\partial \tau_{ji}}{\partial x_j} \tag{6.4}
\]

Substitution of Reynolds decomposition

\[
u_i = \bar{u}_i + u'_i, \quad p = \bar{p} + p', \quad \tau_{ji} = \bar{\tau}_{ji} + \tau'_{ji} \tag{6.5}
\]

into the equation (6.4) results in
\[ \frac{\partial}{\partial t}(\rho \bar{u}_i \bar{u}_k) + \frac{\partial}{\partial t}(\rho u'_i u'_{k}) + \frac{\partial}{\partial x_j}(\rho \bar{u}_i \bar{u}_j \bar{u}_k + \rho u'_i u'_j u'_k + \rho u'_i u'_{k}) = -\bar{u}_i \frac{\partial \bar{p}}{\partial x_k} - u'_i \frac{\partial \bar{p}'}{\partial x_k} \] (6.6)

\[ - \bar{u}_k \frac{\partial \bar{p}}{\partial x_i} - u'_k \frac{\partial \bar{p}'}{\partial x_i} + \bar{u}_i \frac{\partial \bar{r}_{jk}}{\partial x_j} + u'_i \frac{\partial \bar{r}'_{jk}}{\partial x_j} + u'_k \frac{\partial \bar{r}'_{ji}}{\partial x_j} \]

### 6.1.2 Step 2

The \( k \)-th Reynolds averaged Navier Stokes equation

\[ \frac{\partial \rho \bar{u}_k}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{u}_j \bar{u}_k) = -\bar{\partial} \frac{\partial \bar{p}}{\partial x_k} + \frac{\partial}{\partial x_j}(\bar{\tau}_{jk} - \rho u'_i u'_k) \] (6.7)

is multiplied with the \( i \)-th component of averaged velocity

\[ \bar{u}_i \left( \frac{\partial \rho \bar{u}_k}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{u}_j \bar{u}_k) \right) = \bar{u}_i \left( -\bar{\partial} \frac{\partial \bar{p}}{\partial x_k} + \frac{\partial}{\partial x_j}(\bar{\tau}_{jk} - \rho u'_i u'_k) \right) \] (6.8)

Again the \( i \)-th Reynolds averaged Navier Stokes equation is multiplied with the \( k \)-th component of averaged velocity

\[ \bar{u}_k \left( \frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho \bar{u}_j \bar{u}_i) \right) = \bar{u}_k \left( -\bar{\partial} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ji} - \rho u'_i u'_i) \right) \] (6.9)

The sum of two last equations reads

\[ \frac{\partial \rho \bar{u}_i \bar{u}_k}{\partial t} + \frac{\partial \rho \bar{u}_i \bar{u}_k \bar{u}_j}{\partial x_j} = -\bar{u}_i \frac{\partial \bar{p}}{\partial x_k} - \bar{u}_k \frac{\partial \bar{p}}{\partial x_i} + \bar{u}_i \frac{\partial (\bar{\tau}_{jk} - \rho u'_i u'_k)}{\partial x_j} + \bar{u}_k \frac{\partial (\bar{\tau}_{ji} - \rho u'_i u'_i)}{\partial x_j} \] (6.10)

### 6.1.3 Step 3

Subtracting the last equation from (6.6) results in

\[ \frac{\partial}{\partial t}(\rho u'_i u'_{k}) + \frac{\partial}{\partial x_j}(\rho u'_j u'_{k}) + \frac{\partial}{\partial x_j}(\rho u'_i u'_j u'_k) = -u'_i \frac{\partial \bar{p}'}{\partial x_k} - u'_k \frac{\partial \bar{p}'}{\partial x_i} + u'_i \frac{\partial \bar{r}'_{jk}}{\partial x_j} + u'_k \frac{\partial \bar{r}'_{ji}}{\partial x_j} - \rho u'_i u'_k \frac{\partial \bar{u}_j}{\partial x_j} - \rho u'_j u'_k \frac{\partial \bar{u}_i}{\partial x_j} \] (6.11)
Using identities (please prove them)

\[-u_i' \frac{\partial p'}{\partial x_k} - u_k' \frac{\partial p'}{\partial x_i} = p' \left( \frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i} \right) - \left[ \delta_{jk} \frac{\partial}{\partial x_j} (u_i' p') + \delta_{ij} \frac{\partial}{\partial x_j} (u_k' p') \right] \tag{6.12} \]

\[u_i' \frac{\partial \tau_{jk}}{\partial x_j} + u_k' \frac{\partial \tau_{ji}}{\partial x_j} = \mu \left( \frac{\partial^2 u_k'}{\partial x_j^2} + \frac{\partial^2 u_i'}{\partial x_j^2} \right) = \mu \frac{\partial^2}{\partial x_j^2} u_i' u_k' - 2\mu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j} \tag{6.13} \]

we get the Reynolds stress model equation

\[\frac{\partial}{\partial x_j} \left( \rho u_i' u_k' \right) + \frac{\partial}{\partial x_j} (\rho \bar{u}_j u_i' u_k') + \frac{\partial}{\partial x_j} (\rho u_i' u_j u'_k) = \frac{\partial}{\partial x_j} D_{jk} + R_{ik} + P_{ik} - \varepsilon_{ik} \tag{6.14} \]

which can be written in a compact form

\[\frac{\partial}{\partial t} (u_i' u_k') + \bar{u}_j \frac{\partial}{\partial x_j} (u_i' u_k') = \frac{\partial}{\partial x_j} D_{jk} + R_{ik} + P_{ik} - \varepsilon_{ik} \tag{6.14} \]

The physical meaning of terms on the r.h.s is as follows

\[D_{ik} = \nu \frac{\partial (u_i' u_k')}{\partial x_j} - u_i' u_j' u_k' + \frac{1}{\rho} \left( \delta_{jk} u_i' p' + \delta_{ij} u_k' p' \right) \to \text{Diffusion} \tag{6.15} \]

\[R_{ik} = \frac{1}{\rho} \left( \frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i} \right) p' \to \text{Re distribution (energy exchange)} \tag{6.16} \]

\[P_{ik} = -u_j' u_k' \frac{\partial \bar{u}_i}{\partial x_j} - u_j' u_i' \frac{\partial \bar{u}_k}{\partial x_j} \to \text{Generation} \tag{6.17} \]

\[\varepsilon_{ik} = 2\nu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j} \to \text{Dissipation} \tag{6.18} \]
6.1.4 Analysis of terms

The diffusion of energy

\[ D_{ik} = \nu \frac{\partial (u_i'u_k')}{\partial x_j} - \frac{1}{\rho} (\delta_{jk}u_i' + \delta_{ij}u_k')p' \]  

(6.19)

is due to

- molecular diffusion, described by the term:

\[ \nu \frac{\partial (u_i'u_k')}{\partial x_j} \]  

(6.20)

- turbulent diffusion, described by the term:

\[ -u_i'u_j'u_k' \]  

(6.21)

- turbulent diffusion caused by correlation between pressure and velocity fluctuations

\[ \frac{1}{\rho} (\delta_{jk}u_i' + \delta_{ij}u_k')p' \]  

(6.22)

The two last terms are unclosed. Here we face with the famous problem noted first by Friedmann and Keller (1924): Effort to derive the equations for the second order moments results in the necessity of determination of new unclosed terms including third order moments \( u_i'u_j'u_k' \). Using the method proposed by Friedmann and Keller in 1924 it is possible to derive equations for moments of arbitrary order. However, the equation for the \( m - th \) order will contain unclosed moments of the \( m + 1 - th \) order. Impossibility of obtaining of a closed system of equations for a finite number of moments, known as the Friedmann-Keller problem) is a direct consequence of the nonlinearity of the Navier Stokes equations.

The also unclosed term

\[ R_{ik} = \frac{1}{\rho} \left( \frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i} \right) p' \]  

(6.23)

describes the redistribution of the energy between different tensor components \( u_i'u_k' \) caused by correlation between the stresses and pressure fluctuations.

The term
\[ P_{ik} = -u'_j u'_k \frac{\partial \bar{u}_i}{\partial x_j} - u'_j u'_i \frac{\partial \bar{u}_k}{\partial x_j} \]  \hspace{1cm} (6.24)

is responsible for the energy generation, i.e. the transport of the energy transfer from averaged (mean) flow to oscillating flow (fluctuations). And, finally,

\[ \varepsilon_{ik} = 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_k}{\partial x_j} \]  \hspace{1cm} (6.25)

is the dissipation. This unclosed term is responsible for the transformation of the turbulent kinetic energy into the inner energy of the flow.

RSM model based on equations (6.14) is used to determine the Reynolds stresses from the transport equations. It is not based on the Boussinesq approach and takes the anisotropy of stresses into account. This model is the best one among RANS models.
Chapter 7

Equations of the $k - \varepsilon$ Model

7.1 Derivation of the k-Equation

According to definition

$$k = \frac{u'_k u'_k}{2} \quad (7.1)$$

Assuming $i = k$ in the Reynolds stress model equations (6.14)

$$\frac{\partial}{\partial t} (u'_i u'_k) + \bar{u}_j \frac{\partial}{\partial x_j} (u'_i u'_k) = \frac{\partial}{\partial x_j} D_{ik} + R_{ik} + P_{ik} - \varepsilon_{ik} \quad (7.2)$$

ans summing equations for $k=1, 2$ and $3$ we obtain the transport equations for the total kinetic energy $k$:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} D_s + P - \varepsilon_S \quad (7.3)$$

where

$$D_S = \nu \frac{\partial k}{\partial x_j} - \frac{1}{\rho} \delta_{jk} \bar{u}'_k \bar{u}'_j - \bar{u}'_j k', \quad k' = \frac{u'_k u'_k}{2} \quad (7.4)$$

$$P = -\bar{u}'_j u'_k \frac{\partial \bar{u}_k}{\partial x_j} \quad (7.5)$$

$$\varepsilon_S = \nu \frac{\partial u'_k}{\partial x_j} \frac{\partial u'_k}{\partial x_j} \quad (7.6)$$

The relation between the true and pseudodissipation is
\[ \varepsilon = \frac{\nu}{2} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)^2 \approx \varepsilon_S + \frac{\partial}{\partial x_j} \nu \frac{\partial}{\partial x_k} \overline{u'_i u'_k} \]  

(7.7)

For large Reynolds numbers the true dissipation and the pseudodissipation are equal.

\[ \varepsilon \approx \varepsilon_S \]  

(7.8)

More precise analysis shows that

\[ \varepsilon \varepsilon_S \approx Re_t^{-1} \]  

(7.9)

where

\[ Re_t = \sqrt{kL/\nu} \]  

(7.10)

### 7.1.1 Closure of terms of \( k \) equation

Two unknown terms in the diffusion

\[ D_S = \nu \frac{\partial k}{\partial x_j} - \frac{1}{\rho} \delta_{jk} \overline{u'_k p'} - \overline{u'_j k'} \]  

(7.11)

are determined by the gradient assumption

\[ -\frac{1}{\rho} \delta_{jk} \overline{u'_k p'} - \overline{u'_j k'} = \nu_t \frac{\partial k}{\partial x_j} \]  

(7.12)

where \( \sigma_k \) is an empirical constant. Dissipation is determined by the energy containing motion using the formula of Prandtl- Kolmogorov

\[ \varepsilon_s = C_D k^{3/2}/L \]  

(7.13)

The Reynolds stresses are seeking in form proposed by Boussinesq:

\[ -\rho \overline{u'_i u'_j} = \nu_t \left( \frac{\partial \overline{u_j}}{\partial x_i} + \frac{\partial \overline{u_i}}{\partial x_j} \right) - \frac{2}{3} \rho \delta_{ij} k \]  

(7.14)

Substitution of all these approximations into the equation (7.3) results in the \( k \)-Equation

\[ \frac{\partial k}{\partial t} + \overline{u_j} \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + \nu_t \left( \frac{\partial \overline{u_j}}{\partial x_i} + \frac{\partial \overline{u_i}}{\partial x_j} \right) \frac{\partial \overline{u_j}}{\partial x_i} - C_D \frac{k^{3/2}}{L} \]  

(7.15)
7.1.2 Derivation of the $\varepsilon$-Equation

The Navier Stokes equation

$$\frac{\partial u_k}{\partial t} + u_j \frac{\partial u_k}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_k} + \frac{1}{\rho} \frac{\partial \tau_{jk}}{\partial x_j}$$

(7.16)

is differentiated and multiplied with the derivative $\frac{\partial u_i'}{\partial x_k}$

$$\frac{\partial}{\partial x_k} \left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \tau_{ji}}{\partial x_j} \right] \frac{\partial u_i'}{\partial x_k}$$

(7.17)

This gives:

$$\frac{\partial \varepsilon_s}{\partial t} + \bar{u}_j \frac{\partial \varepsilon_s}{\partial x_j} = \frac{\partial}{\partial x_j} D_\varepsilon + P_\varepsilon - \varepsilon_\varepsilon$$

(7.18)

where

$$D_\varepsilon = \nu \frac{\partial \varepsilon_s}{\partial x_j} - u_j' \varepsilon_s' - 2\nu \frac{\partial u_j'}{\partial x_k} \frac{\partial p'}{\partial x_k}$$

(7.19)

$$P_\varepsilon = -2\nu u_j' \frac{\partial u_i'}{\partial x_k} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} - 2\nu \left( \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k} \right)$$

(7.20)

$$\varepsilon_\varepsilon = 2\nu^2 \frac{\partial^2 u_i'}{\partial x_j \partial x_k} \frac{\partial u_i'}{\partial x_k}, \quad \varepsilon_\varepsilon' = \nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}$$

(7.21)

The terms on the r.h.s. were approximated according to the following formula:

$$\frac{\partial}{\partial x_j} D_\varepsilon = \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon_\varepsilon}{\partial x_j} \right] ; P_\varepsilon = \frac{C_{\varepsilon 1}}{k} \bar{\tau}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} ; \varepsilon_\varepsilon = \frac{C_{\varepsilon 2} \varepsilon_\varepsilon^2}{k}$$

(7.22)

Constants are taken from planar jet and mixing layer:

$$C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3$$

(7.23)

Hereby the full closed system of the $k - \varepsilon$ model reads:
\[
\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \left( \nu + \nu_t \right) \frac{\partial k}{\partial x_j} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon \\
\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \right) \frac{\partial \varepsilon}{\partial x_j} \right] + \frac{C_{\varepsilon 1} \varepsilon}{k} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{C_{\varepsilon 2} \varepsilon^2}{k}
\]

Under assumption that the generation of the turbulent energy equals to the its dissipation (the turbulence is in equilibrium, turbulent scales are in the inertial range) Kolmogorov and Prandtl derived the relation between the kinetic energy, the dissipation rate and the integral lengths \( L \):

\[
\varepsilon \approx \frac{k^{3/2}}{L}
\]

From the dimension analysis

\[
\nu_t = C_\mu \sqrt{kL}
\]

follows

\[
\nu_t = C_\mu \sqrt{kL} = C_\mu \frac{k^2}{\varepsilon}
\]

As soon as \( k \) and \( \varepsilon \) are known the turbulent kinematic viscosity \( \nu_t \) is computed from (7.27) and Reynolds stresses can be calculated from the Boussinesq hypothesis and then substituted into the Reynolds averaged Navier Stokes equations. The problem is mathematically closed.

The \( k - \varepsilon \) model is the classical approach, which is very accurate at large Re numbers. At small Re number, for instance close to the wall, the approximations used in derivation of \( k - \varepsilon \) model equations are not valid. To overcome this disadvantage various low Reynolds \( k - \varepsilon \) models were proposed.
Chapter 8

Large Eddy Simulation (LES)

8.1 LES filtering

Within the LES all vortices are subdivided into large resolved vortices and fine subgrid vortices. The border between vortices should lie within the inertial range. The separation of fine scale motions (small fine vortices) from large ones is done using the spatial filtering. Let \( \varphi \) be any stochastic function which is represented as the sum of filtered part and fluctuation:

\[
\varphi = \tilde{\varphi} + \varphi' \quad \tilde{\varphi}(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\vec{x} - \vec{s}, t) F(\vec{s}) d\vec{s}
\]

Here \( F(\vec{s}) \) is the filtering function, satisfying the condition

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{s}) d\vec{s} = 1
\]

Three different filtering functions shown in Fig. 1: ideal filter, Gauss filter and top hat filter. Ideal filter is applied in Fourier space. High frequencies are cut off. Low frequencies are simulated directly. The top hat filter is some kind of smoothing applied in physical space. The simplest case is smoothing over three neighboring points

\[
\tilde{\varphi}_i = \frac{1}{b} (\varphi_{i-1} + a \varphi_i + \varphi_{i+1})
\]

where \( b = 2 + a \).
8.1.1 Properties of filtering

The spatial filtering and Reynolds averaging are both filtering operations. LES spatial filtering has properties which differ from these of Reynolds averaging. First, the spatial averaged quantity is not zero. Double filtering is not equal once filtering.

\[ \varphi' \neq 0, \quad \tilde{\varphi} \neq \tilde{\varphi}, \]

Both conditions are compatible because

\[ \varphi' = \varphi - \tilde{\varphi} = \varphi - \tilde{\varphi} \neq 0 \]

Figure 8.1: Different filtering functions used in LES

Other important properties are similar to these of RANS averaging:
• Averaged sum of two quantities is equal to the sum of averaged quantities:
  \[ \varphi + g = \tilde{\varphi} + \tilde{g} \]

• Filtering operator commutes with the differentiation operator
  \[ \frac{\partial \varphi}{\partial t} = \frac{\partial \tilde{\varphi}}{\partial t}, \quad \frac{\partial \varphi}{\partial x_j} = \frac{\partial \tilde{\varphi}}{\partial x_j} \]

A very important relation which is the consequence of these properties is
  \[ \varphi \phi = \tilde{\varphi} \tilde{\phi} + \tilde{\varphi} \phi' + \tilde{\phi} \phi' + \phi \phi' \]

In the case of Reynolds averaging only the first and the last terms remain.
The properties of large and small scale motions are shown in the table 8.1.

<table>
<thead>
<tr>
<th>Large scale motion</th>
<th>Small scale motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generated by mean flow</td>
<td>Generated by large scale structures</td>
</tr>
<tr>
<td>Depends on the flow geometry</td>
<td>Universal</td>
</tr>
<tr>
<td>regular</td>
<td>Stochastic</td>
</tr>
<tr>
<td>Deterministic description</td>
<td>Stochastic description</td>
</tr>
<tr>
<td>Heterogeneous</td>
<td>Homogeneous</td>
</tr>
<tr>
<td>Anisotrop</td>
<td>Isotrop</td>
</tr>
<tr>
<td>Exists long time</td>
<td>Exists short time</td>
</tr>
<tr>
<td>Diffusive</td>
<td>Dissipative</td>
</tr>
<tr>
<td>Modelling is complicated</td>
<td>easy to model</td>
</tr>
</tbody>
</table>

Table 8.1: Properties of large and small scale motions

A very important conclusion from this table is the fact that the small scale motion is universal. Therefore one can expect that the models describing the small scale motion in contrast to RANS models are also universal.

### 8.2 LES equations

The governing equations of LES are derived from the Navier Stokes equation

\[ \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = \frac{\partial}{\partial x_j} \left[ \rho \nu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] - \frac{\partial p}{\partial x_i} + \rho g_i \quad (8.1) \]

Application of the filter operation to 8.1 results in
\[
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j} \left[ \rho \nu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right] - \frac{\partial \bar{p}}{\partial x_i} + \rho g_i
\]

Averaged sum is equal to the sum of averaged terms:

\[
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j} \left[ \rho \nu \frac{\partial \bar{u}_i}{\partial x_j} \right] - \frac{\partial \bar{p}}{\partial x_i} + \rho g_i \quad (8.2)
\]

Introducing the term\]

\[
\tau_{ij}^{SGS} = u_i u_j - \bar{u}_i \bar{u}_j
\]

The equation 8.2 is rewritten in the final form

\[
\frac{\partial}{\partial t}(\rho \bar{u}_i) + \frac{\partial}{\partial x_j}(\rho \bar{u}_i \bar{u}_j) = \frac{\partial}{\partial x_j} \left[ \rho \nu \frac{\partial \bar{u}_i}{\partial x_j} - \rho \tau_{ij}^{SGS} \right] - \frac{\partial \bar{p}}{\partial x_i} + \rho g_i
\]

The term \( \tau_{ij}^{SGS} = u_i u_j - \bar{u}_i \bar{u}_j \) is the subgrid stress (SGS) which considers the effect of small fine vortices on large scale motion directly resolved on the grid.

### 8.3 Smagorinsky model

Note that the fine scale vortices are not resolved. They filtered out by the filtering operation. The effect of these vortices is taken by the term \( \tau_{ij}^{SGS} \) into account. Since the small vortices are not modeled, the subgrid stress are calculated using phenomenological models. The most recent phenomenological model was proposed by Smagorinsky in 1963. The Smagorinsky model is just the extension of the Boussinesq approach

\[
\tau_{ij} - \frac{1}{3} \tau_{kk} \delta_{ij} \approx -\nu S_{ij},
\]

Smagorinsky introduced the subgrid viscosity \( \nu_S GS \) instead of the turbulent kinematic viscosity

\[
\tau_{ij}^{SGS} - \frac{1}{3} \tau_{kk}^{SGS} \delta_{ij} \approx -\nu_{SGS} S_{ij},
\]

Expression for the subgrid viscosity was obtained by Smagorinsky with the use of idea taken from the Prandtl mixing length theory. According to Prandtl, the turbulent kinematic viscosity is proportional to the mixing length squared and the velocity gradient close to the wall
According to Smagorinsky, the subgrid viscosity is proportional to the magnitude of the strain rate tensor $S_{ij}$ and to a certain length $l_s$ squared

$$\nu_{SGS} = l_S^2 |\tilde{S}_{ij}|, \quad |\tilde{S}_{ij}| = \sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}$$

Where

$$\tilde{S}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_i} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)$$

The length $l_S$ is assumed to be proportional to the mesh size

$$l_S = C_S \Delta$$

where $C_S$ is the constant of Smagorinsky.

The Smagorinsky constant was estimated first by Lilly. The main assumption of the Lilly analysis is the balance between generation

$$P_r = -\tau_{ij}\tilde{S}_{ij} = 2\nu_t \tilde{S}_{ij}\tilde{S}_{ij} = \nu_t |\tilde{S}_{ij}|^2$$

and dissipation of the turbulent kinetic energy

$$\varepsilon = \bar{P} = \nu_t |\tilde{S}_{ij}|^2 = l_S^2 |\tilde{S}_{ij}|^3$$

Lilly estimated the strain rate tensor magnitude for Kolmogorov spectrum

$$\bar{S}^2 \approx 7C\varepsilon^{2/3} \Delta^{-4/3}$$

Substitution of the last formula into 8.3 results in:

$$l_S = \frac{\Delta}{(7C)^{3/4} \left( \frac{\bar{S}^{3/2}}{\bar{S}^3} \right)}$$

Assuming additionally that $\bar{S}^{3/2} \approx \bar{S}^3$, the length $l_S$ and the Smagorinsky constant are expressed through the Kolmogorov constant $C = 1.5$:

$$C_S = \frac{l_S}{\Delta} = \frac{1}{(7C)^{3/4}} \approx 0.17$$

The Smagorinsky constant 0.17 is derived analytically with a few strong assumptions. The experience shows that numerical results agree with measurements much better if a reduced value of the Smagorinsky constant is
used. Common values are 0.065 and 0.1.

Advantages and disadvantages of the Smagorinsky model are summarized in the table 8.2.

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple</td>
<td>Laminar flow is not modelled</td>
</tr>
<tr>
<td>Low computational costs</td>
<td>Constant of Smagorinsky is constant in time and space</td>
</tr>
<tr>
<td>Stable</td>
<td>Actually, the constant is chosen arbitrarily depending on the problem under consideration</td>
</tr>
<tr>
<td>Good accuracy in ideal conditions</td>
<td>Sensible to grid</td>
</tr>
<tr>
<td></td>
<td>Purely dissipativ</td>
</tr>
<tr>
<td></td>
<td>Damping of pulsation is too strong</td>
</tr>
</tbody>
</table>

Table 8.2: Advantages and disadvantages of the Smagorinsky model

We complete this section with a very important comment:

• the LES models are consistent when the resolution increases, i.e. $\Delta \to 0$.

Indeed, if the resolution is increased, $\Delta \to 0$, the SGS stresses disappear. The LES equation is passed to the original Navier Stokes equations. The LES simulation becomes the DNS simulation if $\Delta \to 0$. On the contrary, the URANS simulation is not consistent when $\Delta \to 0$. The Reynolds stresses don’t disappear if the resolution is increased $\Delta \to 0$. The turbulence is then twice resolved.
Chapter 9

Subgrid Stress (SGS) Models

The classical model of Smagorinsky with the parameter $C_S$ being constant for the whole computational domain is proved to be very diffusive. Germano proposed to calculate the Smagorinsky constant $C_S$ being variable both in space and in time, i.e. $C_S = C_S(x, t)$. The constant is determined using the dynamic procedure which is then referred to as the Dynamic Smagorinsky Model (DSM).

9.1 Model of Germano (Dynamic Smagorinsky Model)

According to the definition the subgrid stress is

$$\tau_{ij}^{SGS} = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j$$  \hspace{1cm} (9.1)

Germano introduces the double filtering or the test filtering designated as $\hat{u} = \hat{u}$. Here the tilde symbol means the first filtering with filter width $\Delta$ whereas the hat symbol stands for the second filtering with filter width $\sim 2\Delta$. The symbol means the resulting double filtering. Using the definition (9.1) we can write

$$T_{ij}^{test} = u_i \hat{u}_j - \hat{u}_i \hat{u}_j = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j$$  \hspace{1cm} (9.2)

Filtration of (9.1) results in

$$\hat{\tau}_{ij}^{SGS} = \tilde{u}_i \hat{u}_j - \hat{u}_i \tilde{u}_j$$  \hspace{1cm} (9.3)

Subtracting (9.3) from (9.2) yields
\[ T_{ij} - \tau_{ij} = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j \quad (9.4) \]

We suppose that the double filter width is small. Therefore the Smagorinsky model is valid for both stresses \( \tau_{ij} \) and \( T_{ij} \):

\[ \tau_{ij} - \frac{1}{3} \tau_{kk} \delta_{ij} = -2(C_s \Delta)^2 |\hat{S}_{ij}|(\hat{S}_{ij}) = 2Cm_{ij}, \]

\[ T_{ij} - \frac{1}{3} T_{kk} \delta_{ij} = -2(C_s \hat{\Delta})^2 |\hat{S}_{ij}|(\hat{S}_{ij}) = 2Cm_{ij}, \quad (9.5) \]

The application of the double filter to \( \tau_{ij} \) gives:

\[ \hat{\tau}_{ij} - \frac{1}{3} \hat{\tau}_{kk} \delta_{ij} = 2\hat{C}m_{ij}, \quad (9.6) \]

where \( C = C_s^2 \). Here we supposed that the filtered product of the constant \( C \) with \( m_{ij} \) is equal to the product of filtered \( m_{ij} \) with the same constant \( \hat{C}m \approx Cm \).

We introduce the tensor \( L_{ij} \) which is equal to the difference between test filter and once filtered original SGS stress:

\[ L_{ij} = T_{ij} - \tau_{ij} = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j \quad (9.7) \]

Using this designation we get from (9.5) and (9.6)

\[ 2CM_{ij} = L_{ij} - \frac{1}{3} L_{kk} \delta_{ij} \quad (9.8) \]

where

\[ M_{ij} = m_{ij}^{test} - m_{ij}^{SGS} \]

The system (9.8) is overdefined (six equations for one unknown coefficient \( C \)). To get an unique solution we multiply both the l.h.s. and r.h.s of (9.8) with the tensor \( S_{ij} \). The final result for \( C \) is

\[ C = \frac{L_{ij} \hat{S}_{ij}}{2M_{ij} \hat{S}_{ij}} \quad (9.9) \]

Use of (9.11) is problematic since the denominator \( M_{ij} \hat{S}_{ij} \) can become zero. To overcome this difficulty Lilly proposed to determine the constant from the condition of the minimum residual of the equation (9.8):
\[ Q = \left( L_{ij} - \frac{1}{3} L_{kk} \delta_{ij} - 2CM_{ij} \right)^2 \rightarrow \min \quad (9.10) \]

The minimum is attained at the point with zero derivative of the functional \(Q\) on the parameter \(C\):

\[ \frac{\partial Q}{\partial C} = -4M_{ij} \left( L_{ij} - \frac{1}{3} L_{kk} \delta_{ij} - 2CM_{ij} \right) = 0 \quad (9.11) \]

It follows directly from (9.11):

\[ C = \frac{M_{ij} L_{ij} - \frac{1}{3} L_{kk} \delta_{ij} M_{ij}}{2M_{ij} M_{ij}} = \frac{M_{ij} L_{ij}}{2M_{ij} M_{ij}} \quad (9.12) \]

since \(\delta_{ij} M_{ij} = 0\). The solution (9.12) corresponds to the minimum of \(Q(C)\) since the second derivative \(\partial^2 Q / \partial C^2\) is positive at this point

\[ \frac{\partial^2 Q}{\partial C^2} = 8M_{ij} M_{ij} > 0 \quad (9.13) \]

Theoretically the constant \(C\) can become negative. The case \(C < 0\) and \(\nu_{SGS}\) can be considered as the energy backscattering. However, this leads to strong numerical instability. That is why the dynamic constant is limited from below:

\[ C = \max \left\{ \frac{M_{ij} L_{ij}}{2M_{ij} M_{ij}}, 0 \right\} \geq 0 \quad (9.14) \]

The subgrid kinematic viscosity is always positive

\[ \nu_{SGS} = C \Delta^2 |\tilde{S}_{ij}| \geq 0 \]

### 9.2 Scale similarity models

Despite the fact that diffusion of the classic Smagorinsky model was substantially reduced by the dynamic choice of the Smagorinsky constant, the Dynamic Smagorinsky model remains very diffusive. This disadvantage was overcome within the similarity models. The main point of the similarity model is the assumption that the statistical properties of the once filtered field \(\tilde{u}_i\) are identical to those of the double filtered field \(\tilde{\tilde{u}}_i\). It is the case if the filter width is small. The difference between once and double filtered velocities is negligible, i.e. different scale motions are similar.
Let us consider $\tilde{u}_j$ as the original (unfiltered) field. $\tilde{\tilde{u}}_j$ is the filtered field and $\tilde{u}_j - \tilde{\tilde{u}}_j$ is the pulsation (see Fig. 9.1). Then from the definition of the subgrid stress one obtains

$$\tau_{ij}^{SGS} = \tilde{u}_i \tilde{\tilde{u}}_j - \tilde{\tilde{u}}_i \tilde{u}_j$$  \hspace{1cm} (9.15)$$

The formula (9.15) is the scale similarity model proposed by Bardina et al. [7]. As seen the SGS stress can be calculated directly from the resolved field $\tilde{u}_i$.

### 9.3 Mixed similarity models

The experience shows that diffusion produced by the scale similarity model (9.15) is too low. The numerical calculations are often unstable using this model. Taking the fact into account, that the diffusion of the Smagorinsky model is too high, it was decided to combine the Samgorinsky and scale similarity models to get the proper diffusion. The advantages and disadvantages of both models are summarized as follows

- **Dynamic Smagorinsky Model (DSM):** energy dissipation is overestimated (drawback), energy backscattering is not reproduced (drawback).
- **Scale similarity model:** energy backscattering is reproduced (advantage), energy dissipation is underestimated (drawback).
The Idea is to combine models to strengthen the advantages and to overcome disadvantages of both models. The hybrid model called as the mixed similarity model is written as

$$\tau'_{ij} = (\tilde{u}_i\tilde{u}_j - \tilde{u}_i\tilde{u}_j) - 2(C_S^\Delta \Delta)^2 |\tilde{S}|\tilde{S}_{ij}$$ \hspace{1cm} (9.16)

The mixed model can be derived in a more formal way. For that the velocity decomposition into filtered and pulsation parts:

$$\tilde{u}_i\tilde{u}_j = (\tilde{u}_i + u'_i)(\tilde{u}_j + u'_j) = \tilde{u}_i\tilde{u}_j + u'_i\tilde{u}_j + \tilde{u}_j u'_i + u'_i u'_j$$ \hspace{1cm} (9.17)

is substituted into the SGS stress expression:

$$\tau_{ij}^{SGS} = \tilde{u}_i\tilde{u}_j - \tilde{u}_i\tilde{u}_j$$ \hspace{1cm} (9.18)

Finally we have Leonard's formulation of the mixed model:

$$\tau_{ij}^{SGS} = L_{ij} + C_{ij} + R_{ij}$$ \hspace{1cm} (9.19)

where

- $L_{ij} = \tilde{u}_i\tilde{u}_j - \tilde{u}_i\tilde{u}_j$ is the Leonard stress
- $C_{ij} = \tilde{u}_i u'_j + \tilde{u}_j u'_i$ is the Cross-stress
- $R_{ij} = u'_i u'_j$ is the Reynolds stress

The sum of the cross and Reynolds stresses is calculated via the Smagorinsky model with the dynamically determined constant $C_S$

$$C_{ij} + R_{ij} = -2(C_S^\Delta \Delta)^2 |\tilde{S}|\tilde{S}_{ij}$$ \hspace{1cm} (9.20)

A substantial disadvantage of this formulation is the fact that the Leonard stress does not satisfy the Galilean invariance condition. The Galilean invariance is the independence of basic formula of mechanics on the speed of the reference system.

The classical definition of the SGS stresses possesses the Galilean invariance. Indeed, let $V$ be the speed of the reference system. The velocity relative to the reference system is

$$\vec{W} = \vec{u} + \vec{V}$$ \hspace{1cm} (9.21)

The SGS stress does not depend on the reference system speed:
\[
\tilde{W}_i \tilde{W}_j - \tilde{W}_j \tilde{W}_i = (\tilde{u}_i + V_i + u'_i)(\tilde{u}_j + V_j + u'_j) - (\tilde{u}_i + V_i + u'_i)(\tilde{u}_j + V_j + u'_j) = \\
= (\tilde{u}_i + u'_i)(\tilde{u}_j + u'_j) - (\tilde{u}_i + u'_i)(\tilde{u}_j + u'_j) = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j
\]  

(9.22)

On the contrary, the Leonard stress is not Galilean invariant:

\[
\tilde{W}_i \tilde{W}_j - \tilde{W}_j \tilde{W}_i = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j - V_i \tilde{u}'_j - V_j \tilde{u}'_i
\]  

(9.23)

Germano proposed an alternative formulation

\[
\tau_{ij}^{SGS} = L_{ij}^0 + C_{ij}^0 + R_{ij}^0
\]  

(9.24)

where all stresses are Galilean invariant:

\[
L_{ij}^0 = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j \quad \text{is the Leonard stress}
\]

\[
C_{ij}^0 = \tilde{u}_i u'_j + \tilde{u}'_i \tilde{u}_j - \tilde{u}_i u'_j - \tilde{u}'_i \tilde{u}_j \quad \text{is the Cross stress}
\]

\[
R_{ij}^0 = u'_i u'_j - \tilde{u}_i \tilde{u}_j \quad \text{is the Reynolds stress}
\]

\[
C_{ij}^0 + R_{ij}^0 = -2(C_{ij}^\Delta \Delta)^2 |\tilde{S}| \tilde{S}_{ij}
\]  

(9.25)

**Exercise:** Prove the following facts:

- equivalence of subgrid stresses computed from formulations of Germano (9.24) and the original one proposed by Leonard (9.19),

- Galilean invariance of stresses in Germano’s formulation (9.24).

### 9.3.1 A-posteriori and a-priori tests

Two tests are used to verify LES models. The comon way is the a-posteriori test. The LES simulation is performed and then the flow parameters obtained from the simulation are compared with these from measurement. Depending on comparison results the conclusion about quality of LES models is drawn. The disadvantage of such approach is that the LES results are affected by modelling errors, errors of approximation of differential operators and rounding errors. In a-priori test they can not be separated.
Direct test of quality of subgrid stresses is the a-priori test. First, the subgrid stress is calculated at each time instant from the definition

\[ \tau_{ij}^{SGSdef} = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j \]  

(9.26)

Then the subgrid stress is computed again at each time instant from any model, say Smagorinsky one

\[ \tau_{ij}^{SGSmod} - \frac{1}{3} \tau_{kk}^{SGSmod} \delta_{ij} = -2(C_S \Delta)^2 |\tilde{S}_{ij}| \tilde{S}_{ij} \]  

(9.27)

The subgrid stresses \( \tau_{ij}^{SGSdef} \) and \( \tau_{ij}^{SGSmod} \) averaged in time are compared each with other. If

\[ \tau_{ij}^{SGSdef} \approx \tau_{ij}^{SGSmod} \]

the SGS model is accurate.

A big difficulty of a-priori tests is the determination of velocities \( \tilde{u}_i \). For that it is necessary first to obtain the unfiltered velocities \( u_i \) with spatial and temporal resolutions compared with the Kolmogorov scales. At present this is a big challenge to measure three components of velocity in a volume with high spatial and temporal resolutions. The Particle Image Velocimetry (PIV) measurements are mostly planar measurements within a two dimensional window. Direct Numerical Simulation data are often used as the source for a-priori tests. Three components of velocity in a volume, obtained from DNS, are filtered and utilized for the test. However, it should be noted that DNS simulation is restricted by relatively low Reynolds numbers, whereas main laws of LES are valid for high Re numbers.
Chapter 10

Hybrid URANS-LES methods

10.1 Introduction

As discussed above, the most promising approach to resolve the flow unsteadiness is the Large Eddy Simulation (LES), which is already widely used for research purposes. Typical Reynolds numbers in engineering are very large even at model scales. The grid resolution necessary for a pure LES is so huge that it makes the direct application of LES impossible (see Sec. 10.4). A practical solution of this problem is the use of hybrid URANS-LES methods, where the near body flow region is treated using URANS and far flow regions are treated with LES.

According to Peng [23] the hybrid techniques can be subdivided into flow matching and turbulence matching methods. Within the flow matching methods the interface between URANS and LES is explicitly defined. LES filtered equations are solved in the LES region, whereas URANS equations are solved in the URANS domain. The flow parameters (velocities, kinetic energy) are matched at the interface between the URANS and the LES regions. Among the most important contributions to the development of flow matching methods we mention the works of (Davidson, Dalstroem [13]; Terracol [29]; Jakirlic et. al. [16]; Temmerman et. al. [28]) and others. A serious weakness of this approach is the development of robust procedures to set the URANS-LES interface for complicated flow geometries. Within the framework of the turbulence matching method an universal transport equation is solved in the whole computational domain. The stress terms in this equation are treated in different ways in LES and URANS domains. There are various procedures to distinguish between LES and URANS cells. The most popular hybrid method is Detached Eddy Simulation (DES) proposed
by (Spalart et al. [26]). The original version of this method is based on the classic Smagorinsky LES model and the Spalart-Allmaras (SA) URANS approach. SA is used close to the wall, whereas LES in the rest part of the flow. The switching between the two techniques is smooth and occurs in a "gray" subdomain. There are two major improvements of DES, developed recently. The first one, DDES (Delayed DES), has been proposed to detect the boundary layers and to prolong the RANS mode, even if the wall-parallel grid spacing would normally activate the DES limiter (Spalart [27]). The second one, IDDES (Improved DDES), allows one to solve the problems with modelled-stress depletion and log-layer mismatch. For the details see the review (Spalart [27]). In spite of a wide application area DES has serious principle limitations thoroughly analyzed by (Menter, Egorov [21]). Other versions of the turbulence matching methods using different blending functions to switch the solution between LES and URANS modes were proposed by (Peng [23]; Davidson, Billson [12]; Abe, Miyata [8]) and others.

A very critical point of the turbulence matching methods is the transition from the time (or ensemble) averaged smooth URANS flow to the oscillating LES flow, see (Menter, Egorov, 2005). The oscillations have to appear within a short flow domain in a "gray zone" between LES and URANS. Experience shows that it is extremely difficult to provide a smooth transition of the turbulent kinetic energy passing from the URANS to LES domain. To overcome this problem (Schlueter et al. [25]) and (Benerafa et al. [11]) used an additional forcing term in the Navier Stokes equation artificially enhancing fluctuations in the gray zone. However, the problem of smooth solution transition from URANS to LES still remains as the main challenge for the turbulence matching methods.

10.2 Detached Eddy Simulation (DES)

The most popular hybrid method -detached eddy simulation- was proposed in 1997 by Spalart et al. [26]. The principle of DES is illustrated in Fig. 10.1. Close to the body the solution is calculated using the URANS mode. Far from the wall the LES equations are solved. The grey zone between URANS and LES is the mixed solution.

The classical version of the DES approach is based on the Spalart Almaras (SA) model formulated with respect to the modified turbulent viscosity $\tilde{\nu} =$
\[ \frac{\partial \tilde{\nu}}{\partial t} + \tilde{v}_j \frac{\partial \tilde{\nu}}{\partial x_j} = C_{b1} \tilde{S} \tilde{\nu} - C_w f_w \left( \frac{\tilde{\nu}}{d} \right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial x_k} \left( (\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_k} \right) + \frac{C_{b2}}{\sigma} \frac{\partial \nu}{\partial x_k} \frac{\partial \tilde{\nu}}{\partial x_k} \] (10.1)

where

- \( C_{b1} = 0.1355, \quad C_{b2} = 0.622, \quad C_{\nu 1} = 7.1, \)
- \( \sigma = 2/3, \quad C_{w1} = \frac{C_{b1}}{\kappa^2} + \frac{1 + C_{b2}}{\sigma}, \)
- \( C_{w2} = 0.3, \quad C_{w3} = 2.0, \quad \kappa = 0.41, \)

- \( f_{\nu 1} = \frac{\chi^3}{\chi^3 + C_{\nu 1}^3}, \quad f_{\nu 2} = 1 - \frac{\chi}{1 + \chi f_{\nu 1}}, \)
- \( f_w = g \left[ 1 + \frac{C_{w3}}{g^6 + C_{w3}} \right]^{1/6}, \quad \chi = \tilde{\nu}/\nu, \)
- \( g = r + C_{w2}(r^6 - r), \quad r = \frac{\tilde{\nu}}{S \kappa^2 d^2}, \)
- \( \tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{\nu 2}; \quad S = \sqrt{2\Omega_{ij} \Omega_{ij}}, \)
- \( \Omega_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{\nu}_i}{\partial x_j} - \frac{\partial \tilde{\nu}_j}{\partial x_i} \right) \)
Here $d$ is the distance from the wall. The physical sense of different terms is illustrated in (10.1). Far from the wall the generation and the distruction terms are approaching each to other and the turbulence attains the equilibrium state:

$$\frac{C_{b1}\tilde{S}}{C_{w1}} - f_w\left(\frac{\tilde{\nu}}{d}\right)^2 \sim 0$$

The kinematic viscosity is the calculated from the formula

$$\tilde{\nu} \sim \frac{C_{b1}}{C_{w1}} \tilde{S} d^2$$

which is similar to the Smagorinsky one:

$$\nu_t = l_S|\tilde{S}_{ij}|, \quad |\tilde{S}_{ij}| = \sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}$$

$$l_S = C_S\Delta$$

DES inventors proposed to use the following expression for $d$:

$$d = \min\{d, C_{DES}\Delta\}, \quad \Delta = \max\{\Delta x, \Delta y, \Delta z\}$$

where $C_{DES} \approx 1.3$ is the DES constant. Now the main idea of the DES becomes obvious:

- At small wall distance $d < C_{DES}\Delta$ the Spalart Almaras URANS model is active
- At large wall distance $d > C_{DES}\Delta$ the Spalart Almaras URANS model is smoothly passed into the Smagorinsky model.

Samples of DES applications are presented in Fig. 10.2 and 10.3.

Despite of the wide application Detached Eddy Simulation technique is not free of disadvantages. Menter [21] notes: The essential concern with DES is that it does not continuously change from RANS to LES under grid refinement. In order for LES structures to appear, the grid spacing and time step have to be refined beyond a case-dependent critical limit. In addition, a sufficiently large instability mechanism has to be present to allow the rapid formation of turbulent structures in regions where the DES limiter is activated. If one of the two, or both requirements are violated, the resulting model is undefined and the outcome is largely unpredictable.
10.3 Description of the hybrid model proposed in Rostock

Our hybrid model is based on the observation that the basic transport equations have the same form in LES and RANS

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_i \bar{u}_j)}{\partial x_j} = -\frac{\partial \bar{p}^*}{\partial x_i} + \frac{\partial (\tau^l_{ij} + \tau^t_{ij})}{\partial x_j},$$

(10.2)

but the interpretation of the overline differs. In LES it means filtering, but in RANS it stands for the Reynolds, or ensemble, averaging. Here we used the standard notation of $p^*$ for the pseudo-pressure, and $\tau^l_{ij}$ and $\tau^t_{ij}$ for the laminar and turbulent stresses respectively. Note that the turbulent stresses are calculated in different ways in LES and URANS regions.
The computational domain in our model is dynamically (i.e. at each time step) divided into the LES and URANS regions. A cell of the mesh belongs to one or the other region depending on the relation between the integral length scale $L$ and the extended LES filter $\Delta$ according to the following rule:

\[
\begin{align*}
\text{if } L > \Delta & \text{ then the cell is in the LES region,} \\
\text{if } L < \Delta & \text{ then the cell is in the URANS region.}
\end{align*}
\] (10.3)

The integral length scale is calculated from the known formula of Kolmogorov and Prandtl with the correction factor 0.168 taken from [24]

\[
L = C \frac{k^{3/2}}{\varepsilon},
\] (10.4)

where $k$ is the turbulent kinetic energy and $\varepsilon$ is the dissipation rate. The constant $C$ is $C \sim 0.168$ close to the wall $y/\delta < 0.2$, $C \sim 0.35$ at $0.2 < y/\delta < 0.7$ and $C \sim 1.0$ in the outer area of the boundary layer $y/\delta > 0.7$, where $\delta$ is the boundary layer thickness. $L$ varies from one time step to another, which results in varying decomposition of the computational domain into the LES and URANS regions. The extended LES filter is computed as

\[
\Delta = \sqrt{d_{\text{max}}^2 + \delta^2},
\] (10.5)

where $d_{\text{max}}$ is the maximal length of the cell edges $d_{\text{max}} = \max(d_x, d_y, d_z)$ and $\delta = (\text{the cell volume})^{1/3}$ is the common filter width used in LES. This choice ensures that very flat cells in the boundary layer (for which $\delta \approx 0$ but $d_{\text{max}} > 0$) are treated correctly. $\Delta$ depends only on the mesh and it is precomputed only once before the main computation.
The LES and URANS regions are shown in Fig. 10.4. The URANS region is located close to the ship surface and plays the role of a dynamic wall function. In areas of bilge vortices formation, the boundary layer is shedding from the hull and penetrates into the outer flow part. Since the boundary layer is a fine scale flow the procedure (10.3) recognizes the bilge vortex formation zones as URANS ones. There is a technical issue concerning the cells which are far from the ship hull and where both $k$ and $\varepsilon$ are small, so large numerical errors are introduced into the integral length scale computed according to Eq. (10.4). To avoid an irregular distribution of URANS and LES zones, the general rule (10.3) of the domain decomposition is corrected in such a way that the LES region is switched to URANS one if $k$ is getting less than some threshold. This procedure has no influence on the ship flow parameters since it is used far from the area of the primary interest.

We have performed several calculations with different combinations of LES and URANS models to find the most efficient one for the problem under consideration. Among the models we used in our computations are the linear and nonlinear $k-\varepsilon$, $k-\omega$ SST and $k\varepsilon v^2 f$ URANS models combined with the simple and dynamic Smagorinsky as well as with the dynamic mixed LES closure models. The experience shows that the most satisfactory results are obtained using the URANS approach based on the $k\varepsilon v^2 f$ turbulent model of [14] and LES approach based on the Smagorinsky dynamic model. The turbulent stresses $\tau_{ij}$ are calculated from the Boussinesq approximation using the concept of the turbulent viscosity. The only difference between LES and URANS is the definition of the kinematic viscosity. Within LES it is considered as the subgrid viscosity and calculated according to the dynamic model of Smagorinsky:

$$\nu_{SGS} = c_D \delta^2 |S_{ij}|, \quad S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

(10.6)

where $S_{ij}$ is the strain velocity tensor and $c_D$ is the dynamic constant. In the URANS region the viscosity is calculated from the turbulent model of [14]:

$$\nu_t = \min \left( 0.09 \frac{k^2}{\varepsilon}, 0.22 \nu^2 T_t \right),$$

(10.7)

where $\nu^2$ is the wall normal component of the stresses and $T_t$ is the turbulent time $T_t = \max(k/\varepsilon, 6\sqrt{\nu/\varepsilon})$. 109
10.4 Sample of hybrid method application for the tanker KVLCC2

The hybrid model described in the previous section has been implemented in the open-source code OpenFOAM based on the finite volume method. The doubled model of the KRISO tanker KVLCC2 (Kim et al.[18]) with the scale 1/58 has been chosen for our investigations since it is a well-known benchmark which is widely used in the shipbuilding community (Goetheburg workshop [15]). The model has length of 5.517 m, breadth of 1 m, draught of 0.359 m and block coefficient of 0.8098. The computations have been carried out on two unstructured 3D-grids. The coarse grid contains $1.2 \times 10^6$ cells, the fine one — $1.8 \times 10^6$ cells. Both grids are coarse in the foreship area ($y^+_i \approx 10$). The grid with $1.2 \times 10^6$ cells has $y^+_i \approx 2 - 6$ in the wall region of aftership. The grid with $1.8 \times 10^6$ cells was additionally refined in the aftership area and has $y^+_i \approx 0.1 - 4$ in the wall region. This grids were generated by the Ship Model Basin Potsdam (SVA Potsdam) and proved to be an appropriate grid for RANS calculations. The computations on the both grids have been carried out with fixed maximal Courant number of 0.6. The time step was about 0.0008 s for the coarse grid and 0.0005 s for the fine one.

For the space discretization, the central difference scheme is used for all terms in the momentum equation whereas the Crank-Nicholson scheme is used for the time discretizations. Steady RANS solutions are used to initialize the flow in the computational domain. The time-averaged solutions has been obtained when the resolved flow reached a statistically steady state (usually it requires the ship way of $3 - 4$ lengths). A typical time period for statistical averaging takes about $40 - 50$ seconds, which corresponds to $8 - 10$ lengths of the ship way. Study of the wake has been performed for the constant velocity of 1.047 m/s corresponding to the Reynolds number of $Re = 4.6 \times 10^6$. The Froude number $Fn = 0.142$ is small which makes it possible to neglect the water surface deformation effects.

Estimations of the resolution necessary for a pure ship LES

The precise determination of the necessary LES resolution is quite difficult. Estimations presented below are based on the idea that about 80% of the turbulent kinetic energy should be directly resolved and the rest is modeled in a properly resolved LES simulation. Implementation of this idea implies the knowledge of the Kolmogorov $\eta$ and the integral length $L$ scales which are
used to draw the typical spectra of the full developed turbulence $E(k)$. The wave number $k^*$ separating the resolved and modeled turbulence is found from the condition

$$\frac{\int_{k^*}^{\infty} E(k) dk}{\int_{0}^{\infty} E(k) dk} \approx 0.2.$$  

(10.8)

The maximum possible cell size is then $\Delta_{\text{max}} = 2\pi/k^*$. The scales $L$ and $\eta$ are found from the known expression $\eta = (\nu^3/\varepsilon)^{1/4}$ and Eq. (10.4), where the kinetic energy $k$ and the dissipation rate $\varepsilon$ are taken from RANS simulations using $k$-$\varepsilon$ linear model. The ratio $\lambda = \Delta_{\text{max}}/\eta$ is then used as the scale parameter for grid generation. Both lengths vary in space which makes the grid generation procedure very complicated. To roughly estimate the size of the grid we assume that $\lambda$ is constant. We performed different calculations determining $\lambda$ at the two following points: i) the point where $L/\eta$ is maximal in the boundary layer and ii) the point in the propeller disk where the vorticity $\vec{\omega}$ is maximal (region of the concentrated vortex structure). The latter is dictated by the wish to resolve the most intensive vortex flow structures which have the strongest influence on the propeller operation. Since LES application is required in the ship stern area only this part of the computational volume has been meshed. It covers the boundary layer of the stern region starting from the end of the parallel midship section. The thickness of the meshed region has been constant and equal to the maximum boundary layer thickness at the stern $\delta_{BL}$. The grid for a pure LES is generated using the following algorithm. The minimum Kolmogorov length $\eta_{\text{min}}$ is determined in the near wall region. The cell sizes in $x$ and $z$ directions along the wall are calculated by multiplication of $\eta_{\text{min}}$ with the scale parameter $\lambda$. These sizes remain constant for all cells row in $y$ direction which is normal to the ship surface (see Fig. 10.5). The cells have at least two equal sizes which is desirable from the point of view of LES accuracy. The choice of the size in $y$ direction is dictated by proper resolution of the boundary layer. Close to the wall this size is chosen from the condition $\Delta_w = \min(y_w, \eta_{\text{min}})$. Since $y_w$ is chosen as the ordinate where $y^+=1$ the first nodes lay deeply in the viscous sublayer. The size in $y$ direction at the upper border of the boundary layer is equal to $\Delta_{\infty} = \lambda \eta_\delta$, where $\eta_\delta$ is the Kolmogorov scale at $y = \delta_{BL}$. A simple grading is used in $y$ direction between $\Delta_w$ and $\Delta_{\infty}$.

Results of the estimations are as follows: the required grid size ranges from $\sim 5M$ to $\sim 25M$ for $\text{Re} = 2.8 \times 10^6$, and from $\sim 7M$ to $\sim 60M$ for $\text{Re} = 5.8 \times 10^6$. The results vary depending on the value of $\lambda$ in use, so they should be considered as very rough estimations. Together with simi-
lar estimations for the nonlinear $k-\varepsilon$ model these results show that the LES grid should have the order of tens of millions of nodes. Nowadays, the computations with hundred millions and even with a few billions of nodes are becoming available in the research community. However, a numerical study of engineering problems implies usually many computations which have to be performed within a reasonable time with moderate computational resources. In this sense, the results of the present subsection clearly demonstrate that the pure LES is impossible for ship applications so far. To verify that the resolution estimation procedure we used gives meaningful results, it has been applied for turbulent boundary layer (TBL) benchmark. We found from methodical calculations that the pure LES with 1M cells is quite accurate for prediction of the velocity distribution, TBL thickness, TBL displacement thickness and the wall shear stress. The estimation procedure presented above predicted the necessary resolution around 0.5M. Therefore, the estimations presented for a ship model are rather lower bound for the resolution required for a pure LES.

In the CFD community one can observe tendency to use pure LES without paying any attention to resolution problems. Very often LES is running on typical RANS grids. In fact, such computations can give correct results if the flow structures to be captured are large enough and exist for a long time. In some cases modeling of such structures does not require detailed resolution of boundary layers and a thorough treatment of separation regions. As an example one can mention flows around bluff bodies with predefined separation lines like ship superstructures. Application of underresolved LES for well streamlined hulls should be considered with a great care. First of all, one should not forget that the basic LES subgrid models are derived under the assumption that at least the inertial turbulent subrange is resolved. Second, underresolution of wall region leads to a very inaccurate modeling of the boundary layer, prediction of the separation and overall ship resistance. It is clearly illustrated in the Table 10.1. The resistance obtained from underresolved LES using the wall function of (Werner, Wengle [30]) is less than half of the measured one and that obtained from RANS. Obviously, the application of modern turbulence LES models, more advanced than RANS models, does not improve but even makes the results much worse with the same space resolution. The change from RANS to LES should definitely be followed by the increase of the resolution which results in a drastic increase of the computational costs. These facts underline necessity of further development towards hybrid methodology. Although in (Alin et al. [9]) it has been shown that the accuracy of the resistance prediction using pure LES at a very moderate resolution with $y^+ \sim 30$ can be improved using special wall functions, the
most universal way for the present, to our opinion, is application of hybrid methods.

10.4.1 Validation

<table>
<thead>
<tr>
<th>Method</th>
<th>$C_R$ ($10^{-3}$)</th>
<th>$C_P$ (%)</th>
<th>$C_F$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>KRISO Exp.</td>
<td>4.11</td>
<td>15</td>
<td>85</td>
</tr>
<tr>
<td>RANS $k\varepsilon v^2 f$</td>
<td>4.00</td>
<td>16</td>
<td>84</td>
</tr>
<tr>
<td>$k-\omega$ SST SAS</td>
<td>3.80</td>
<td>18</td>
<td>82</td>
</tr>
<tr>
<td>Underresolved LES</td>
<td>1.70</td>
<td>81</td>
<td>19</td>
</tr>
<tr>
<td>Hybrid RANS LES</td>
<td>4.07</td>
<td>17</td>
<td>83</td>
</tr>
</tbody>
</table>

Table 10.1: Results of the resistance prediction using different methods. $C_R$ is the resistance coefficient, $C_P$ is the pressure resistance and $C_F$ is the friction resistance.

Before we start to analyze unsteady effects, we show that our hybrid method predicts averaged flows with the accuracy not worse than that of RANS. Table 10.1 confirms that the hybrid method works well for ship resistance prediction. Both the overall resistance and the resistance components ratio are in a good agreement with the KRISO measurements (Kim et al. [18]) and RANS. Disagreement between LES results and measurements is due to coarse resolution of the boundary layer.

The axial mean velocity field in the propeller plane for the KVLCC2 shown
in Fig. 10.6(a) is compared with the experimental data of KRISO (Lee et al. 2003). The axial velocity $u_x$ is normalized to the ship model velocity $u_0$ and the coordinates are normalized to the length between perpendiculars of the ship model $L_{pp}$. The mean velocity field is very similar to the experimental one. The lines of the constant velocity have the typical form and reflect the formation of a large longitudinal bilge vortex in the propeller disk. The second longitudinal vortex is formed near the water plane, but it has a much smaller strength compared to the bilge one. More detailed comparison is given in Fig. 10.7. The axial mean velocity in circumferential direction at the propeller radii $r/R = 0.7$ and $r/R = 1$ is compared with KRISO experimental data and simulations using the $k$-$\omega$ SST-SAS and hybrid models. The agreement in the ranges $0 < \theta < 40^\circ$ and $90^\circ < \theta < 180^\circ$ can be considered as quite satisfactory. The results for the range $40^\circ < \theta < 90^\circ$ require additional numerical investigations and measurements. However, bearing in mind a big scattering of experimental data in this range, the observed discrepancy is not necessary the sign of modeling weakness. Fig. 10.8 shows the resolved turbulent kinetic energy normalized to $u_0^2$ for both computational grids in comparison with experimental data of KRISO. Topologically, the isolines are similar to those of the axial mean velocity shown in the Fig. 10.6(a). Refinement of the grid leads to a significant improvement of the numerical results. The position of the area with the strongest fluctuations and the magnitudes of these fluctuations are reproduced better when the resolution increases. According to (Lee at al.[20]), the uncertainty of the measured TKE is $\sim 12\%$, so our results are quite satisfactory. The OpenFOAM implementation of the detached eddy simulation approach DDES failed to predict the averaged velocity field properly, see Fig. 10.6(b). Advanced URANS technique $k$-$\omega$ SST-SAS provides quite satisfactory results, see Fig. 10.6(c) and Fig. 10.7.
Figure 10.6: The mean axial velocity field $u_x/u_0$ in the propeller plane computed with different models (right) vs. measurements (left).

(a) Hybrid URANS-LES.

(b) SA-DDES.

(c) $k$-$\omega$-SST-SAS.

Figure 10.7: Circumferential distribution of the mean axial velocity field in the propeller plane. $\triangle$ — $k$-$\omega$-SST-SAS, $\bigcirc$ — DSM+V2F, solid line — KRISO experiments for the specified $r/R$.

(a) $r/R = 0.7$

(b) $r/R = 1.0$

115
Figure 10.8: Resolved turbulent kinetic energy $k = \rho/2(u'_xu'_x + u'_yu'_y + u'_zu'_z)/u'_0^2$ multiplied with $10^3$ in the propeller plane. Numerics (right-half of each figure) versus measurement (left-half).

Figure 10.9: Positions of probe points. $R$ is the propeller radius.
Figure 10.10:
Bibliography


